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ONE-WAY FLOW NETWORK FORMATION UNDER CONSTRAINTS

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One-way flow network formation under constraints*

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Abstract

We study the effects of institutional constraints on stability and efficiency in the "one-way flow" model of network formation. In this model the information that flows through a link between two players runs only towards the player that initiates and supports the link, so in order for it to flow in both directions both players must pay whatever the unit cost of a directional link is. We assume that an exogenous "societal cover" consisting of a collection of possibly overlapping subsets covering the set of players specifies the social organization in different groups or "societies", so that a player may initiate links only with players that belong to at least one society that he/she also belongs to, thus restricting the feasible strategies and networks. In this setting, we examine the impact of such societal constraints on stable/efficient architectures and on dynamics.

JEL Classification Numbers: A14, C72, D20, J00

Key words: Network formation, One-way flow model, Stability, Dynamics.

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1 Introduction

In a seminal paper Bala and Goyal (2000) provide two benchmark non cooperative models of network formation. In both models links are formed unilaterally and the network allows information or other benefits to flow through it. In the "one-way flow" model the information flows through a link between two players only in the direction of the player that initiates and supports the link, so in order for it to flow in both directions both players must pay whatever the unit cost of a directional link is. In the "two-way flow" model the information flows through a link between two players in both directions irrespective of who pays for it[1]. In both settings, Bala and Goyal study Nash and strict Nash stability and provide a dynamic model, first assuming that information flows without friction and then dropping this assumption. In both models the current network is assumed to be common knowledge to all players, who may unrestrictedly initiate links with any other players. These authors show that, when no friction exists, stability in the sense of Nash equilibrium is equivalent to minimal connectedness in either model, while in the stronger sense of *strict* Nash stability "wheels" are the only stable architectures in the one-way flow model and "center-sponsored stars" are the only stable architectures in the two-way flow model.

These benchmark models have been extended since then in different directions[2]. In Olaizola and Valenciano (2011) we argue that: *"Due to what is generically referred to here as "institutional constraints" (social, cultural, linguistic, geographical, economic, etc.), individuals may often see only "part of the world" and initiate links only within that part or a part of that part. Thus, it seems more realistic to assume that a set of possibly overlapping groups (family, tribe, clan, club, gender, age, linguistic community, nationality, professional association, department, etc., depending on the context) configures the social constraints within which individuals interact. More precisely, we assume that each individual may initiate links only within the groups he/she belongs to."* In that paper we address the same issues as Bala and Goyal (2000), but assume some institutional or social constraints, namely a "societal cover" consisting of a collection of groups of players called "societies" that covers the whole set of players is exogenously given and it is assumed that each individual can only establish links with players with whom he/she shares membership of at least one society[3]. In Olaizola and Valenciano (2011) this study is conducted for the two-way flow model only. In the absence of decay, the strict Nash stable architectures are characterized and proved to exist for any societal cover and to be highly hierarchical in their organization: they form

[1] A third benchmark model is Jackson and Wolinsky's (1996), where the formation of a link between two players requires the agreement of both.

[2] There is a growing body of literature on these extensions that we decline to summarize here. Some recent books surveying part of this literature are Goyal (2007), Jackson (2008) and Vega-Redondo (2007). See also Jackson's (2010) survey.

[3] In Olaizola and Valenciano (2011) it is proved that the societal cover model provides the most general *symmetric* link-formation constraint that can be considered, i.e., any such constraint can be interpreted as associated with a societal cover.

oriented trees, perhaps "grafted", that collapse into Bala and Goyal's center-sponsored stars when the societal cover consists of a single society. Bala and Goyal's dynamic model is also applied to this extension and proved to converge at least in payoffs.

In this paper we address a similar study of link formation under institutional constraints *for the one-way flow model*, establishing some positive and negative results. We first prove equivalence between Nash stability and minimal connectedness. We then characterize strict Nash networks consistent with a societal cover as minimally connected networks formed by interconnected wheels where these interconnections satisfy a qualifying condition. This characterization is used to establish some features of strict Nash networks and, more importantly, to establish *their possible non existence*. Nevertheless, we prove their existence for some simple types of societal cover. Finally, we study Bala and Goyal's dynamics for the one-way flow model in this setting. Here we also have negative results, showing the *possibility of non convergence* even in payoffs. Nevertheless, we prove convergence for the types of societal cover for which we have proved the existence of strict Nash networks.

The paper closest to this one is Galeotti (2006), where equilibrium in the one-way flow model is studied in the presence of heterogeneity in costs and benefits[4]. Nevertheless, although our societal cover means a form of heterogeneity, Galeotti's model does not include, and is not included in, the one we study here. But, as is shown later, both models can be seen as particular cases of a general model with heterogeneity about which no result has been obtained as far as we know.

The rest of the paper is organized as follows. In Section 2, the basic model is specified and the necessary notation and terminology are given. Section 3 studies stability and efficiency under institutional constraints. In Section 4, Bala and Goyal's dynamic model is applied to this setting. Finally, Section 5 summarizes the main conclusions.

2 The model: one-way flow under constraints

We describe formally the ingredients of the model by recalling some definitions from Bala and Goyal (2000) and Olaizola and Valenciano (2011).

Let $N = \{1, 2, .., n\}$ denote the set of *nodes* or *players*. Each player may choose other players with whom to initiate and support *links*. By $g_{ij} \in \{0, 1\}$ we denote the existence ($g_{ij} = 1$) or not ($g_{ij} = 0$) of a link connecting i and j initiated by i, and when such a link exists we refer to it as "link $\overleftarrow{ij}$". Vector $g_i = (g_{ij})_{j \in N \setminus i} \in \{0, 1\}^{N \setminus i}$ specifies[5] the set of links supported by i and is referred to as an (unrestricted) *strategy* of player i. $G_i := \{0, 1\}^{N \setminus i}$ denotes the set of i's (unrestricted) strategies and $G_N = G_1 \times G_2 \times .. \times G_n$ the set of (unrestricted) strategy profiles. An unrestricted strategy profile $g \in G_N$

[4] Other papers dealing with heterogeneity in the one-way flow model are Billand et al. (2008), Derks and Tennekes (2009), and Derks et al. (2009).

[5] We always drop the brackets "{..}" in expressions such as $N \setminus \{i\}$.

univocally determines a directed *N-network* (N, Γ_g), where

$$\Gamma_g := \{(i,j) \in N \times N : g_{ij} = 1\},$$

which we identify with g and refer to as network g. If $M \subseteq N$ we denote by $g\mid_M$ the *M-network* $(M, \Gamma_{g|_M})$ with

$$\Gamma_{g|_M} := \{(i,j) \in M \times M : g_{ij} = 1\},$$

which we refer to as the *M-subnetwork* of g. As N is usually clear from the context, we generally write just "network" instead "N-network".

We assume that an exogenous "societal cover" consisting of a set of possibly overlapping "societies" imposes a social constraint: each player in N can initiate links only with players with whom he/she shares membership of at least one society. Formally, we have the following

Definition 1 *(Olaizola and Valenciano, 2011) A "societal cover" of N is a collection of subsets of N (called "societies"), $\mathcal{K} \subseteq 2^N$, such that: (i) $\bigcup_{A \in \mathcal{K}} A = N$, and (ii) for all $A, B \in \mathcal{K}$ $(A \neq B)$, $A \nsubseteq B$.*

Thus, every player belongs to at least one society and no society contains any other.

We denote by $\mathcal{K}_i \subseteq \mathcal{K}$ the *affiliation* of player i or set of *societies* to which i belongs, and by $N(\mathcal{K}_i) \subseteq N$ player i's *reach,* i.e., the set of *nodes* that i may directly access, that is:

$$\mathcal{K}_i := \{A \in \mathcal{K} : i \in A\}$$

and

$$N(\mathcal{K}_i) := \bigcup_{A \in \mathcal{K}_i} A.$$

A *component $\mathcal{C}$ of a societal cover $\mathcal{K}$* is a subset $\mathcal{C} \subseteq \mathcal{K}$ such that *(i)* for all $A, B \in \mathcal{C}$ there exist $A_1, .., A_k \in \mathcal{K}$ s.t. $A_1 = A$ and $B = A_k$, and $A_i \cap A_{i+1} \neq \varnothing$ for $i = 1, .., k-1$, and *(ii)* for all $B \in \mathcal{K} \backslash \mathcal{C}$, $B \cap (\cup_{A \in \mathcal{C}} A) = \varnothing$. The subset $\cup_{A \in \mathcal{C}} A$ of N covered by a component $\mathcal{C}$ is denoted by $N(\mathcal{C})$. A societal cover is *connected* if it has a unique component.

The following definition constrains the structure of a network so as to be consistent with a given societal cover of N.

Definition 2 *(Olaizola and Valenciano, 2011) A network g is consistent with a societal cover $\mathcal{K}$ (or is a $\mathcal{K}$-network) if for every link $g_{ij} = 1$ there exists an $A \in \mathcal{K}$ s.t. $i, j \in A$ (i.e., $\mathcal{K}_i \cap \mathcal{K}_j \neq \varnothing$).*

A vector $g_i = (g_{ij})_{j \in N(\mathcal{K}_i) \backslash i} \in \{0,1\}^{N(\mathcal{K}_i) \backslash i}$ specifies a set of $\mathcal{K}$-*feasible* links initiated by i and is referred to as a *$\mathcal{K}$-admissible strategy* of player i. $G_i(\mathcal{K}) := \{0,1\}^{N(\mathcal{K}_i) \backslash i}$ denotes the set of i's $\mathcal{K}$-admissible strategies and $G_{\mathcal{K}} = G_1(\mathcal{K}) \times G_2(\mathcal{K}) \times .. \times G_n(\mathcal{K})$

the set of $\mathcal{K}$-admissible strategy profiles. A $\mathcal{K}$-admissible strategy profile g determines a $\mathcal{K}$-network that is identified with g.

We say that there is a *path of length k from j to i* in g if there exist $k+1$ players $j_0, j_1, .., j_k$, s.t. $i = j_0$, $j = j_k$, and for all $l = 1, .., k$, $g_{j_{l-1}j_l} = 1$. When such a path exists we write $j \xrightarrow{g} i$. Note that the *direction* of a path is crucial: a path from j to i allows information or benefits to travel from j to i, but *not* from i to j. The set of players with whom i supports a link is denoted by $N^d(i; g)$, and the set of players connected with i by a path (union $\{i\}$) by $N(i; g)$, and their cardinalities by $\mu_i^d(g) := \#N^d(i; g)$ and $\mu_i(g) := \#N(i; g)$.

A *component* of a network g is a subnetwork $g \mid_C$, where $C \subseteq N$, such that for any two players j, i in C, $j \xrightarrow{g} i$, and no set strictly containing C meets this condition. We say that g is *connected* if g is the unique component of g. A component of a network is *minimal* if for all i, j s.t. $g_{ij} = 1$, the number of components of g is smaller than the number of components of $g - ij$, where $g - ij$ is the network that results from replacing $g_{ij} = 1$ by $g_{ij} = 0$ in g. A network is *minimally connected* if it is connected and minimal.

We denote by g_{-i} the network where all links supported by i in g are deleted, and by (g_{-i}, g'_i) the strategy profile and network that results from replacing g_i by g'_i in g.

It is assumed that each node contains a valuable and particular type of information and a link g_{ij} allows such information to flow from j to i, without friction or decay, so that each node i receives the information from all nodes with which it is connected by a path, i.e., all j such that $j \xrightarrow{g} i$. Let $v_{ij} > 0$ be the payoff that player i derives from connecting directly (by a link) or indirectly (by a path) with player j, and $c_{ij} > 0$ the cost for player i of initiating a link with j. Thus, the payoff of player i in g is

$$\Pi_i(g) = \sum_{j \in N(i;g)} v_{ij} - \sum_{j \in N^d(i;g)} c_{ij}.$$

We assume costs and benefits to be homogeneous across players (i.e., $v_{ij} = v$ and $c_{ij} = c$, for all i, j)[6]. We also assume $v > c$, so that connecting with *new* nodes (i.e., those with which one node is not connected by a path) is always profitable. In short, we assume:

$$\Pi_i(g) = v\mu_i(g) - c\mu_i^d(g) \qquad (v > c). \tag{1}$$

A $\mathcal{K}$-network is *efficient* if it maximizes the aggregate payoff under the constraint of $\mathcal{K}$-feasible payoffs, that is, those that can be obtained by means of $\mathcal{K}$-networks.

[6] In fact, the constraint imposed by a societal cover means a form of heterogeneity that could be formulated in terms of payoffs. For this assume that each agent i has a cost

$$c_{ij} \begin{cases} = c, & \text{if } j \in N(\mathcal{K}_i) \\ = M, & \text{otherwise;} \end{cases}$$

where M is a sufficiently large number, and $v_{ij} = v$ for all j. Note the difference with the other forms of heterogeneity that, as far as we know, have been considered in the literature.

3 Stability and efficiency

The following definitions are natural extensions of the notions of Nash stability and strict Nash stability, following Bala and Goyal (2000), for a network in a scenario where payoffs are given by (1) and: (i) a societal cover $\mathcal{K}$ allows only for links connecting individuals that have at least one society in common, and (ii) all players in the same component $\mathcal{C}$ of $\mathcal{K}$, i.e., in $N(\mathcal{C})$, have common knowledge of the part of the current network connecting individuals of $N(\mathcal{C})$. Condition (ii) can be justified by assuming that information about which is the current network propagates between overlapping societies[7]. Note that this scenario yields the unconstrained environment of Bala and Goyal (2000) for the particular case of the simplest societal cover: $\mathcal{K} = \{N\}$.

Definition 3 *A Nash $\mathcal{K}$-network is a $\mathcal{K}$-network g that is stable under $\mathcal{K}$-admissible strategies, that is, for all $i \in N$:*

$$\Pi_i(g) \geq \Pi_i(g_{-i}, g'_i) \quad \text{for all } g'_i \in G_i(\mathcal{K}). \tag{2}$$

When (2) holds, we say that g_i is a *best (admissible) response* of i to g_{-i}. The stability notion can be refined in the strict sense:

Definition 4 *A strict Nash $\mathcal{K}$-network is a Nash $\mathcal{K}$-network g such that for all $i \in N$:*

$$\Pi_i(g) > \Pi_i(g_{-i}, g'_i) \quad \text{for all } g'_i \in G_i(\mathcal{K}) \ (g'_i \neq g_i). \tag{3}$$

Thus, (3) means that in a strict Nash $\mathcal{K}$-network every player is playing his/her *unique* best (admissible) response to those played by the others. Note that for $\mathcal{K} = \{N\}$ a (strict) Nash $\mathcal{K}$-network is a (strict) Nash network in the standard setting.

Note that the set of players in each component of the societal cover form a separate world: no link with players in other components is possible and no information about them reaches them. In particular the following straightforward result follows easily.

Proposition 1 *A $\mathcal{K}$-network g is a Nash (strict Nash) $\mathcal{K}$-network if and only if $g\mid_{N(\mathcal{C})}$ is a Nash (strict Nash) $\mathcal{C}$-network for each component $\mathcal{C}$ of $\mathcal{K}$.*

Remark: In view of Proposition 1 and the irrelevance of societies consisting of a single individual, in what follows *our attention is constrained to connected societal covers where all societies contain at least two individuals.*

The following proposition extends Bala and Goyal's result to this setting.

[7] As a set, each society can be conceived as a *complete* network connecting all its members that allows for a certain level of information to flow, including the current $\mathcal{K}$-network but *not* the information that flows through it. There is no conflict with the interpretation of the model if we assume that the $\mathcal{K}$-network we consider now, with associated costs and benefits, allows for the flow of a particular type of information that cannot flow through such basic underlying complete societal network.

Proposition 2 *Given a connected societal cover $\mathcal{K}$ of N, a $\mathcal{K}$-network g is a Nash $\mathcal{K}$-network if and only if it is minimally connected.*

Proof. *Necessity* ($\Rightarrow$): Let $\mathcal{K}$ be a connected societal cover of N, and g a $\mathcal{K}$-network. Assume g is not connected. Then there exist two nodes $i, j \in N$ such that there is no path from j to i in g ($j \overset{g}{\nrightarrow} i$). As cover $\mathcal{K}$ is connected, a finite sequence of nodes $i_1, .., i_m$ exists, such that $i_1 = i$, $i_m = j$ and for each $k = 1, .., m-1$, there is an $A \in \mathcal{K}$ s.t. $i_k, i_{k+1} \in A$. Then for at least two consecutive nodes among these m nodes, say i_k and i_{k+1}, there is no path in g from i_{k+1} to i_k. But then it is feasible and profitable for node i_{k+1} to initiate a link with i_k. Thus g must be connected. If g were not minimal there would be some superfluous link that could be eliminated and that would benefit the player that eliminated it, and consequently g would not be a Nash $\mathcal{K}$-network.

Sufficiency ($\Leftarrow$): Reciprocally, assume that g is minimally connected. Let i be any player and g'_i be any strategy $g'_i \in G_i(\mathcal{K})$ ($g'_i \neq g_i$). We show that $\Pi_i(g) \geq \Pi_i(g_{-i}, g'_i)$. If node i does not support any link then g is not minimally connected. If i supports only one link then $\Pi_i(g) \geq \Pi_i(g_{-i}, g'_i)$ for all $g'_i \in G_i(\mathcal{K})$. Then assume that i supports at least two links. A new strategy $g'_i \neq g_i$ means deleting some links and initiating new ones. If the number of links initiated is greater than or equal to the number of deleted ones, i's payoff cannot increase. Assume then that i deletes a number of links and replaces them by a strictly lower number of links. As g is minimally connected, each deleted link was necessary to connect i with at least one node. Then at least for one of the new links, say $\overleftarrow{ij}$, j must be connected by a path with two of the nodes, say k and k', that have been disconnected from i due to i's deletion of links, otherwise the payoff of i would decrease in g'. As g is minimally connected there exist paths $j \overset{g}{\rightarrow} i$, $k \overset{g}{\rightarrow} j$ and $k' \overset{g}{\rightarrow} j$. If $j \overset{g}{\rightarrow} i$ contains the link $\overleftarrow{ik}$, then $\overleftarrow{ik'}$ was superfluous for i in g. Similarly, if $j \overset{g}{\rightarrow} i$ contains the link $\overleftarrow{ik'}$, then $\overleftarrow{ik}$ was superfluous for i in g. Finally, if $j \overset{g}{\rightarrow} i$ contains neither $\overleftarrow{ik}$ nor $\overleftarrow{ik'}$, then both were superfluous for i in g. Thus in all three cases g could not be minimally connected. ∎

Remark: Minimal connectedness is a necessary condition for efficiency, but, by contrast with the two-way flow model of Bala and Goyal (2000) and its extension in Olaizola and Valenciano (2011), in the one-way flow model a $\mathcal{K}$-network may be minimally connected but not efficient. In Figure 1 two minimally connected $\mathcal{K}$-networks are represented[8]: (a) is an efficient Nash $\mathcal{K}$-network, while (b) is a Nash $\mathcal{K}$-network that is not efficient.

We now concentrate on *strict* Nash $\mathcal{K}$-networks. Bala and Goyal (2000) prove that in their single-society setting wheels are the only strict Nash architectures. In their setting a wheel is a sequence of players that *includes all* and where each player supports a (unique) link with the next one in the sequence and the last one with the first, or,

[8] As in all figures, nodes are represented by dots (without labels unless convenient), and links by arrows between them with the convention that the node at the tip of the arrow supports it.

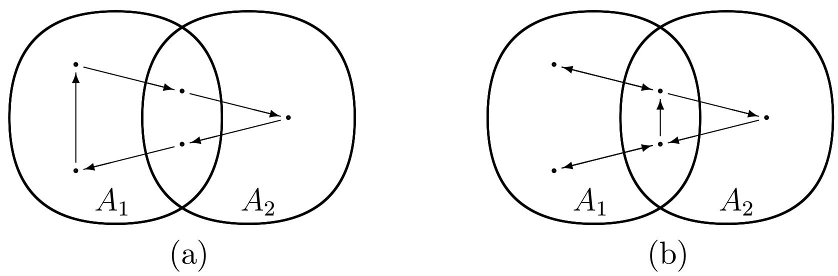

Figure 1: Efficient and non-efficient Nash $\mathcal{K}$-networks

equivalently, a minimally connected network where each player supports exactly one link. In the context of $\mathcal{K}$-networks a wheel in the sense of Bala and Goyal may not even be feasible for certain covers, but, as we show below, wheels are also important in connection with strict Nash $\mathcal{K}$-networks. This motivates the following definition.

Definition 5 *A set of players $M \subseteq N$ ($\#M \geq 2$) is said to be connected by a wheel w in a network g if $g\mid_M = w$ and w is a minimally connected M-network where each player supports exactly one link.*

Note that according to this definition: (i) a wheel does *not* necessarily connect all players in N; (ii) a node in a wheel can link, or be linked from, other nodes different from those in the wheel. When $M = N$ we say that the wheel is *all-encompassing*.

Re-stated in terms of the current setting, notation and terminology, and adapted to it, Bala and Goyal (2000) establish the following result: *when $\mathcal{K} = \{N\}$, i.e., the cover consists of a single society, the only strict Nash $\mathcal{K}$-networks are all-encompassing wheels*[9].

As we show below, the societal cover diversifies the strictly stable networks. A variety of structures of interconnected wheels emerges as possible strict Nash $\mathcal{K}$-networks depending on the structure of the societal cover; moreover, in general, several architectures appear as strict Nash for a given societal cover. Our next goal is to identify and characterize these networks. As a first result, we have a characterization that will allow us to establish some salient features of strict Nash $\mathcal{K}$-networks and prove their *existence or non-existence under certain conditions.* Namely, we have the following characterization: strict Nash $\mathcal{K}$-networks are minimally connected networks consisting of wheels, which satisfy a qualifying condition that restricts the way in which those wheels can interconnect.

Theorem 1 *A network g is a strict Nash $\mathcal{K}$-network if and only if the following three conditions hold: (i) g is a minimally connected $\mathcal{K}$-network, (ii) g is formed by wheels, and (iii) if $w(j)$ denotes a wheel containing j and for any i not contained in $w(j)$ it holds $g_{ij} = 1$, then for all k s.t. $N(k; g - ij) = N$, $k \notin N(\mathcal{K}_i)$.*

[9] Given their weaker assumptions on payoffs, the empty network may also be strict Nash in their setting.

Proof. *Necessity* ($\Rightarrow$): Let g be a strict Nash $\mathcal{K}$-network. Condition (i): By Proposition 2, g is minimally connected. Condition (ii): Let i be any node. As g is connected, there is some node j such that $i \overset{g}{\to} j$ and $j \overset{g}{\to} i$. Thus there is a cycle, i.e., a sequence of nodes $i_1, .., i_m$ such that $i_1 = i_m = i$ and $g_{i_l i_{l+1}} = 1$. We can assume that i only appears at the ends of this sequence (if i appears somewhere else in this sequence just take a subsequence where this condition holds). Now if no node appears twice in the cycle, as g is minimally connected, the nodes in the cycle are connected by a wheel in g (i.e., unless the cycle contains only two nodes, no two nodes in the cycle are mutually linked) and we are done. Otherwise, delete the subcycles until a wheel is finally obtained. Condition (iii): Assume conditions (i) and (ii) hold, but for any $i \notin w(j)$ it holds $g_{ij} = 1$ and for some k s.t. $N(k; g - ij) = N$, it is $k \in N(\mathcal{K}_i)$. Then i can replace the link with j by a link with k and keep his/her payoff.

Sufficiency ($\Leftarrow$): Let g be a $\mathcal{K}$-network satisfying conditions (i)-(iii). Let i be any node. By condition (i) i supports at least one link and cannot delete any without loss. We prove that i cannot replace any link without loss either. Assume $g_{ij} = 1$. Denote by $W(i)$ the set of wheels containing i in g. If $W(j) \subseteq W(i)$, then there is no $k \neq i$ s.t. $g_{kj} = 1$ (otherwise j would belong to a wheel to which i would not belong) and then i could not replace g_{ij} by any other link and remain connected with j. Finally, if $W(j) \nsubseteq W(i)$ then for some wheel containing j, $w(j)$, $i \notin w(j)$, so that condition (iii) applies and i cannot replace link g_{ij} and keep his/her payoff. ■

As an immediate consequence the following is obtained.

Corollary 1 *Any all-encompassing wheel that is a $\mathcal{K}$-network is a strict Nash $\mathcal{K}$-network. In particular, when $\mathcal{K} = \{N\}$ the only strict Nash $\mathcal{K}$-networks are all-encompassing wheels.*

Now based on the characterization provided by Theorem 1, we establish some salient features of the architecture of strict Nash $\mathcal{K}$-networks. The first one gives a necessary condition for the contact and existence of *two* wheels in strict Nash $\mathcal{K}$-networks: the existence of "hinge-players", that is, unique nodes in the intersection of the reaches of another two.

Proposition 3 *In a strict Nash $\mathcal{K}$-network g if a node i is linked by two different players j and k, then i is the unique node in the intersection of j's reach and k's reach, that is, $N(\mathcal{K}_j) \cap N(\mathcal{K}_k) = \{i\}$.*

Proof. Let i, j, k be three nodes in a strict Nash $\mathcal{K}$-network g s.t. $g_{ji} = g_{ki} = 1$. Assume there is an $i' \neq i$ that is within j's and k's reach. As g is minimally connected, $i \overset{g}{\to} i'$, i.e., there is a path from i to i'. Now if link $\overleftarrow{ji}$ is not on the path $i \overset{g}{\to} i'$, then j can replace it by $\overleftarrow{ji'}$ without loss. Similarly, if link $\overleftarrow{ki}$ is not on the path $i \overset{g}{\to} i'$, then k can replace it by $\overleftarrow{ki'}$ without loss. Otherwise, assume that both links $\overleftarrow{ji}$ and $\overleftarrow{ki}$ are on the path $i \xrightarrow{g} i'$. Without loss of generality assume that link $\overleftarrow{ji}$ goes first on that path, then j can replace $\overleftarrow{ji}$ by $\overleftarrow{ji'}$ without loss. ■

As a corollary, there are two conclusions: a sufficient condition under which the only possible architecture of a strict Nash $\mathcal{K}$-network is the all-encompassing wheel; and a fact: within a society information may converge at a node but never diverge in a strict Nash $\mathcal{K}$-network.

Corollary 2 *(i) If no two societies in a cover $\mathcal{K}$ intersect in a singleton, a strict Nash $\mathcal{K}$-network is necessarily an all-encompassing wheel. (ii) Within a society, information may converge at a node from another two, but never diverge in a strict Nash $\mathcal{K}$-network.*

Proof. (i) By Theorem 1, a strict Nash $\mathcal{K}$-network is formed by wheels, and the existence of two or more wheels means necessarily that some node is linked by two different players. By the preceding proposition, such a node must be the unique node at the intersection of the reaches of these two, or, equivalently, any two societies containing this node and either of the other two must share only that node. Now if no two societies in cover $\mathcal{K}$ intersect in a singleton this is not possible, thus only an all-encompassing wheel can be a strict Nash $\mathcal{K}$-network.

(ii) By the preceding proposition, no three nodes i, j, k s.t. $g_{ji} = g_{ki} = 1$ in a strict Nash $\mathcal{K}$-network g, can belong to the same society. ■

In Olaizola and Valenciano (2011) the characterization of strict Nash $\mathcal{K}$-networks for the two-way flow model under the constraints imposed by a societal cover allows an easy constructive proof of the existence of such networks. In contrast with this result, the question of existence in the one-way flow model is answered *negatively* by the preceding consequences of the characterization and the following example.

Example 1: Let $N = \{1, 2, 3, 4, 5\}$ and let the cover $\mathcal{K} = \{\{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}$. The *non existence of a strict Nash $\mathcal{K}$-network* for this cover can be derived independently from Corollary 2 and from Proposition 3. Given that, as the reader can easily check, no all-encompassing wheel is feasible for this cover non existence follows from Corollary 2. But this conclusion also follows directly from Proposition 3: in order to be connected, players $1, 2$ and 3 must link to either 4 or 5, but then necessarily two of them link to the same player and whoever they are their reaches share *two* players.

This negative result raises the question of conditions for a cover $\mathcal{K}$ under which a strict Nash $\mathcal{K}$-network does exist. In general, each particular cover is a case-study. We now describe some simple "regular" covers for which the existence of strict Nash $\mathcal{K}$-networks is guaranteed.

Consider a *linear* societal cover of the form $\mathcal{K} = \{A_i\}_{i=1,2,..,m}$, where for all $i = 1, 2, .., m-1$, $A_i \cap A_{i+1} \neq \varnothing$, and in all other cases two societies do not intersect. First consider the case where for all $i = 1, 2, .., m-1$, $\#(A_i \cap A_{i+1}) > 1$. In this case the all-encompassing wheel is $\mathcal{K}$-feasible and therefore strict Nash $\mathcal{K}$-networks exist (see Fig. 2 (a)). Moreover, in view of Corollary 2-i, this is the only possible architecture of a strict Nash $\mathcal{K}$-network. Now consider the case where for some i, $\#(A_i \cap A_{i+1}) = 1$. In this case the all-encompassing wheel is not feasible, but a strict Nash $\mathcal{K}$-network can be constructed by using wheels to connect nodes in any subsequence of consecutive societies whose intersection contains more than one node. Then these wheels interconnect by means of the nodes at those intersections that contain a single node, thus forming a strict Nash $\mathcal{K}$-network (see Fig. 2 (b)).

Consider a cover *in wheel* of the form $\mathcal{K} = \{A_i\}_{i=1,2,..,m}$ s.t. for all $i = 1, 2, .., m-1$, $A_i \cap A_{i+1} \neq \varnothing$, and $A_1 \cap A_m \neq \varnothing$, and in all other cases two societies do not intersect. For such covers the all-encompassing wheel is $\mathcal{K}$-feasible and therefore strict Nash. Just pick m nodes $i_1, .., i_m$, with $i_j \in A_j \cap A_{j+1}$, for $i = 1, 2, .., m-1$, and $i_m \in A_1 \cap A_m$, then starting at i_1, connect all unconnected nodes within society A_j forming a path $i_j \to i_{j+1}$ till a wheel is completed (see Fig. 2 (c)). Thus, by Corollary 2-i, when all intersections contain at least two nodes the all-encompassing wheel is the only architecture of a strict Nash $\mathcal{K}$-network; while when some intersection contains a single node, architectures other than the all-encompassing wheel are also feasible for a strict Nash $\mathcal{K}$-network (see Fig. 2 (d)).

The *societal core* (Olaizola and Valenciano, 2011) of a cover $\mathcal{K}$ is the set of nodes that belong to all societies, i.e., $core(\mathcal{K}) := \cap_{A \in \mathcal{K}} A$. Then the following conclusion can be drawn. When the core of a cover $\mathcal{K}$ is not empty and contains at least as many nodes as the number of societies in $\mathcal{K}$, then the all-encompassing wheel is $\mathcal{K}$-feasible and therefore strict Nash. Just pick as many nodes in the core as there are societies in the cover, say m, $\{i_1, .., i_m\} \subset core(\mathcal{K})$. Starting at i_1 connect all unconnected nodes within $A_j \setminus \{i_{j+2}, .., i_m\}$ forming a path $i_j \to i_{j+1}$ till a wheel is completed (see Fig. 2 (e)). Observe that this condition does not hold in Example 1, where no strict Nash $\mathcal{K}$-network exists, but note also that this sufficient condition is not necessary (see, e.g., Fig. 2 (f)).

4 Dynamics

We now apply Bala and Goyal's (2000) dynamic model in this setting. Namely, starting from any initial $\mathcal{K}$-network g, in each period each player i with a positive probability responds with a $\mathcal{K}$-admissible best response to g_{-i} (this includes any strategy that yields the same payoff to i as the current one when no strategy can improve i's payoff), or randomizes across them when there are more than one. Otherwise, player i exhibits *inertia*, i.e., keeps his/her links unchanged. In this way, a Markov chain on the state space of all $\mathcal{K}$-networks is defined. Bala and Goyal prove that in their setting, i.e., for $\mathcal{K} = \{N\}$, starting from any network, the dynamic process converges to a strict Nash network with probability 1. In other words, the only absorbing sets are singletons consisting of wheels.

Olaizola and Valenciano (2011) shows how Bala and Goyal's dynamic model for the two-way flow model may fail to converge to a strict Nash $\mathcal{K}$-network when a societal cover constrains link-formation. This possibility leads to the introduction of the notion

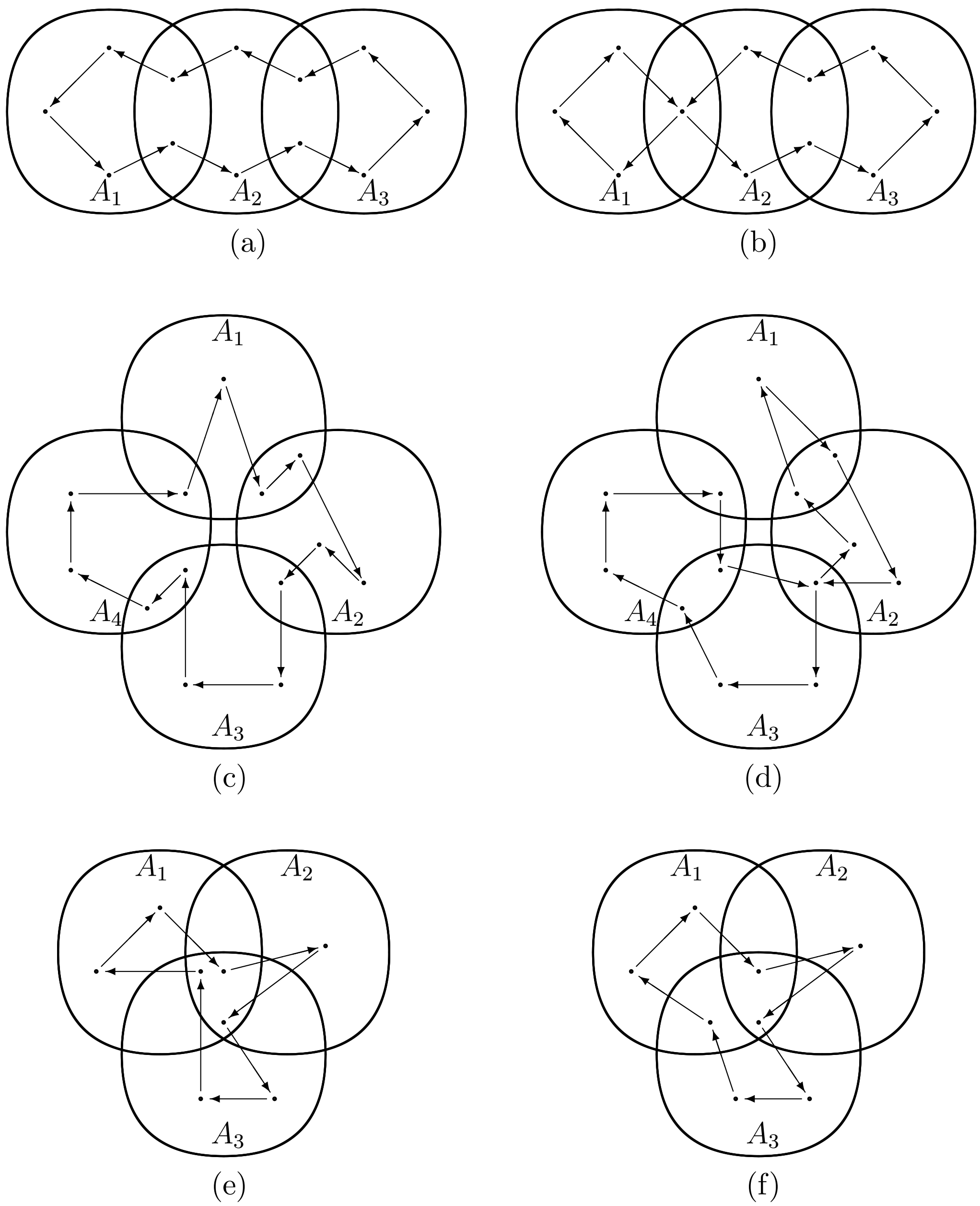

Figure 2: Strict Nash $\mathcal{K}$-networks

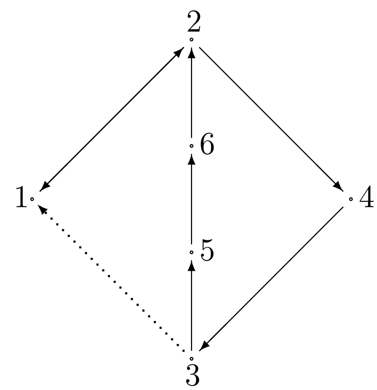

Figure 3: Example 2

of *quasi* strict Nash $\mathcal{K}$-networks, constituents of the absorbing sets for such dynamics, whose existence is proved. These *quasi* strict Nash $\mathcal{K}$-networks are just minimally connected $\mathcal{K}$-networks which are "miscoordination-proof", i.e., such that they cannot be disconnected by best response dynamics, and such that when the dynamic reaches one of them the payoffs remain stable for all players in spite of everlasting oscillations. The following example shows such a situation for the one-way flow model:

Example 2: Let $N = \{1, 2, 3, 4, 5, 6\}$ and let $\mathcal{K}$ be the cover $\mathcal{K} = \{\{1, 2\}, \{1, 3\}, \{3, 5, 6\}, \{2, 3, 4, 6\}\}$. Consider the $\mathcal{K}$-network g such that $g_{12} = g_{21} = g_{26} = g_{65} = g_{53} = g_{34} = g_{42} = 1$, and $g_{ij} = 0$ otherwise. Network g is represented in Figure 3. Given the affiliation of player 1, his/her only best response consists of replacing link $\overleftarrow{12}$ by $\overleftarrow{13}$, while no other player has a best response. In the network that results if 1 replaces link $\overleftarrow{12}$ by $\overleftarrow{13}$, player 1's best response is to replace $\overleftarrow{13}$ by $\overleftarrow{12}$, and no other player has a best response (for instance, player 5 *without constraints* could replace $\overleftarrow{53}$ by $\overleftarrow{54}$ or $\overleftarrow{51}$, but these links are not admissible given 5's affiliation). In fact, the best response dynamic would oscillate between these two networks, with player 1 alternatively linking 2 and 3. Note that every player receives the same payoff in either of these networks.

One might expect a similar result here to the one obtained for the two-way flow model: convergence of the dynamic process to strict or quasi-strict Nash $\mathcal{K}$-networks. Once again another difference with the two-way flow model is encountered: in the one-way flow model it may be the case, as is for certain covers, that no miscoordination-proof network exists. It can be checked that this is the case for Example 1: for this societal cover, starting from any $\mathcal{K}$-network, best response would keep oscillating like a kaleidoscope, connecting and disconnecting the network and never reaching a strict Nash or a quasi-strict Nash $\mathcal{K}$-network, which means in particular that *convergence is not even guarantee in terms of payoffs.* With this general result about convergence discarded, each societal cover is a case-study. Although the issue of convergence when strict Nash $\mathcal{K}$-networks exist remains open, we have established it for the tree types of cover for which we show the existence of strict Nash $\mathcal{K}$-networks.

Proposition 4 *Let $\mathcal{K}$ be a societal cover, if $\mathcal{K}$ is (i) linear or (ii) in wheel or (iii) its core contains at least as many nodes as there are societies in $\mathcal{K}$, then Bala and Goyal's best response dynamic model converges to a strict Nash $\mathcal{K}$-network, with probability* 1.

The proof consists of an adaptation of Bala and Goyal's proof of their Theorem 3.1, which requires all the complications that a societal cover entails to be overcome. It is given in the Appendix and illustrates how these complications can, at least in these cases, be circumvented.

5 Concluding remarks

Olaizola and Valenciano (2011) studies the impact of institutional constraints, as modeled by a societal cover, on Bala and Goyal's (2000) benchmark two-way flow model. This paper addresses a similar study for their other benchmark model: the one-way flow model. The table below summarizes the main results, stressing both the parallelisms and the significant differences in the models in the context of link-formation constrained by a societal cover. If center-sponsored stars as strict Nash networks generalize to oriented trees (perhaps "grafted" when there are "hinge-players", i.e., individuals who are the only one at the intersection of the reaches of some other two) in the context of $\mathcal{K}$-networks, wheels must generalize to possibly interconnected wheels in this context when there are such hinge-players. But, in contrast with the two-way flow model, in the one-way flow model strict Nash $\mathcal{K}$-networks *may not exist*. Finally, Bala and Goyal's dynamic model, which in the two-way flow model never fails to converge to a strict or quasi-strict Nash $\mathcal{K}$-network, may in the one-way flow model fail to converge to a quasi-strict Nash $\mathcal{K}$-network, thus failing to converge even in terms of payoffs. In short, the impact of introducing the constraint of a societal cover seems to be greater on the results for the one-way flow model than for the two-way flow model.

$\mathcal{K}$-network	**Two-way flow model**	**One-way flow model**
Nash	= minimally connected	= minimally connected
Efficient	Efficient = Nash	Nash *not* always efficient
Strict Nash (S.N.)	always exist	may *not* exist
S. N. architectures	oriented (grafted) trees	(interconnected) wheel(s)
Hinge players	necessary for grafted	necessary for multiple wheels
Dynamics' convergence	always to S.N./quasi-S.N.	may *not* converge

Needless to say, if such difficulties arise when no friction is assumed in the flow of information through the network, one can only expect further difficulties with the introduction of decay. In fact, in the presence of decay we have not been able to obtain any useful results.

Appendix

Proof of Proposition 4: (i) Consider first the case of a linear cover with two societies, $\mathcal{K} = \{A, B\}$, whose intersection is non-empty. We prove that starting from any $\mathcal{K}$-network g there is a positive probability of transiting to a strict Nash $\mathcal{K}$-network (absorbing state in the Markov process) in finite time. As there is a positive probability at each period that all but one agent will exhibit inertia, it suffices to see that starting from any $\mathcal{K}$-network there is a finite sequence of players' best responses that leads to a strict Nash $\mathcal{K}$-network. To that end, take a player i_1 in $A \backslash B$, let him/her play a best response and let g^1 denote the resulting $\mathcal{K}$-network where i_1 observes at least all nodes in A.

First step: Form a wheel containing all nodes in A.
Take a node i_2 furthest away in A from i_1 in g^1 (i.e., the/a node in A for which the length of the shortest path $i_2 \overset{g^1}{\to} i_1$ is the greatest). This means that i_1 observes all nodes in A without using any of i_2's links, otherwise a node other than i_2 would be furthest away in A from i_1. There is then a best response of i_2 where $\overleftarrow{i_2 i_1}$ is the only link with a node in A (if $i_2 \in A \cap B$ such a best response may include some other links with nodes in $B \backslash A$). Let i_2 play that best response (or "play inertia" if it was his/her current strategy [10]), and let g^2 denote the resulting $\mathcal{K}$-network. Note that in g^2 node i_1 still observes all nodes in A. Now we describe the induction step from a current network g^k and a sequence of nodes in A, $i_1, i_2, .., i_k$, such that each node i_r $(r = 2, .., k)$ supports with i_{r-1} the only link with nodes in A, and i_1 observes all nodes in A. Let i_{k+1} be the node furthest away in A from i_k in g^k. Again, this means that i_{k+1} observes all nodes in A without using any of i_k's links. There is then a best response of i_{k+1} in which his/her only link with a node in A is the link with i_k, and i_1 still observes all nodes in A. Repeat until $\{i_0, i_1, i_2, .., i_k\} = A$. At this stage agent i_1 observes all nodes in $A \cup B$ and he/she is the only node in A with possibly more than one link, then a best response of his/her is to form only one link with i_k and delete all others. At the end of this all nodes in A form a wheel and observe all nodes in $A \cup B$.

Second step: Form an "8" consisting of this wheel containing all nodes in A and another one containing all nodes in $B \backslash A$ and one node in $A \cap B$.
Take a node j_1 in $A \cap B$ that is linked by a node in $A \backslash B$ (at least one must exist). Let j_2 be the node in $B \backslash A$ furthest away from j_1. Let j_2 play the best response consisting of forming a single link with j_1. By reiterating this process a sequence $j_1, j_2, .., j_l$ is formed where $\{j_2, .., j_l\} \subset B \backslash A$ and each node j_r $(r = 2, .., l)$ supports the only link with j_{r-1}, and j_1 still observes all nodes in $A \cup B$. Repeat until $\{j_2, .., j_l\} = B \backslash A$. At this stage agent j_1 is the only one with possibly more than one link with nodes in $B \backslash A$ and a best response of his/her is to form a single link in $B \backslash A$ with j_l and keep the one with a node in $A \backslash B$. At the end of this we have a wheel containing all nodes in $B \backslash A$ and node j_1. Therefore this wheel and the one formed in the previous step form

[10] In what follows we omit this clause as obvious when a player plays a best response.

the desired "8". If $A \cap B = \{j_1\}$ the network obtained is a strict Nash $\mathcal{K}$-network, otherwise:

Third step: We show that a sequence of best responses leads to two wheels, one containing all nodes in A, the other containing all nodes in B and *both sharing a sequence of links containing all those in* $A \cap B$.

Remember that j_1 was followed (i.e., linked) by a node in $A \backslash B$. Let i_r be the first node after j_1 in the sequence in $A \cap B$, and let $i_{r'}$ be the first node after i_r in this sequence that is followed (i.e., linked) by a node in $A \backslash B$. Now we describe a sequence of best responses: first, j_2 deletes his/her link with j_1 and links with $i_{r'}$; second, the player in $A \backslash B$ that links $i_{r'}$ deletes this link and links with the node in $A \backslash B$ that is linked by i_r; third, let i_r delete his/her link and link j_1; fourth, let the player linking with j_1 delete his/her link and link with $i_{r'}$. After these four best response movements, the reader may check that we have two wheels, one containing all nodes in $B \backslash A$, the other containing all those in A, and both sharing the sequence $j_1, i_r, .., i_{r'}$, where each of these nodes links with the preceding one. If this sequence contains all players in $A \cap B$, we have the desired pair of wheels, otherwise, repeat these four steps starting at $i_{r'}$ instead of at j_1. By reiterating this process we obtain a pair of wheels as desired.

Fourth step: Form an all-encompassing wheel.

Relabel by $k_1, .., k_m$, the sequence consisting of all nodes in $A \cap B$, where each of these nodes links with the preceding one. Again we give a sequence of best responses. Let j be the player in $B \backslash A$ who links k_m. Let j replace his/her link with k_m by a link with k_1. We again have an "8" in which two wheels interconnect at k_1, but now all nodes in $A \cap B$ are consecutively linked. This allows the following sequence of best responses. Let k_2 replace his/her link with k_1 by a link with the player, say j', in $B \backslash A$ linked by k_1. Then k_1 can delete his/her link with j'. Now k_3 replaces his/her link with k_2 by a link with j', and subsequently k_2 can delete his/her link with j'. Reiterate this till k_m replaces the link with k_{m-1} by a link with j', and k_{m-1} deletes his/her link with j'. At this stage an all-encompassing wheel is formed.

Now consider a *linear* societal cover of the form $\mathcal{K} = \{A_i\}_{i=1,2,..,m}$, where for all $i = 1, 2, .., m-1$, $A_i \cap A_{i+1} \neq \varnothing$, and in all other cases two societies do not intersect. Then start at A_1, take a node in $A_1 \backslash A_2$ and form a wheel containing all nodes in A_1 proceeding as in the first step. Then, take a node in $A_1 \cap A_2$ that is linked by a node in $A_1 \backslash A_2$ and, proceeding as in the second step, form an "8" consisting of the wheel containing all nodes in A_1 and another containing all nodes in $A_2 \backslash A_1$ and the node chosen in $A_1 \cap A_2$. Then, following steps 3 and 4 (unless there is a unique node in $A_1 \cap A_2$), form a wheel containing all nodes in $A_1 \cup A_2$. At this stage all nodes in this wheel observe at least all nodes in $A_1 \cup A_2 \cup A_3$. Iterate this process, now taking a node in $A_2 \cap A_3$, etc., until an all-encompassing wheel is completed or a sequence of wheels (in this case contacting at isolated nodes at the intersection of two consecutive societies) forming a strict Nash $\mathcal{K}$-network.

(ii) Consider a cover *in wheel* of the form $\mathcal{K} = \{A_i\}_{i=1,2,..,m}$ s.t. for all $i = 1, 2, .., m-1$, $A_i \cap A_{i+1} \neq \varnothing$, and $A_1 \cap A_m \neq \varnothing$, and in all other cases two societies do not intersect.

Now proceed as in the linear case starting at any society, say A_r. Take a player i_1 in $A_r \backslash A_{r-1}$. Let him/her play a best response. Now i_1 observes at least all nodes in A_r. Form a sequence of nodes in A_r as described inductively. Let $i_1, i_2, .., i_k$ $(k \geq 1)$ be the sequence formed up to step k. Let i_{k+1} be the node furthest away in A_r from i_k and let i_{k+1} play a best response in which his/her only link in A_r is the link with i_k, *if such a best response exists*[11], and reiterate this step as far as such a best response exists until a wheel containing all nodes in A_r is formed. Otherwise (i.e., if at some point before the wheel is formed i_{k+1} has no such best response), let him/her play any best response. This best response should include a link with a node, say i'_1, either in $A_{r-1} \backslash A_r$ or $A_{r+1} \backslash A_r$. Now recommence a sequence starting at i'_1 from the current network. It can be seen that by reiterating this process a wheel will be formed including all nodes in a society[12], unless a strict Nash $\mathcal{K}$-network is formed in the process. Without loss of generality assume that the wheel has been formed in A_1. Now proceed as in step 2, by picking a node j_1 in $A_1 \cap A_2$ linked by a node in $A_1 \backslash A_2$ and forming a sequence of nodes in $A_2 \backslash A_1$ as described inductively. Let $j_1, j_2, .., j_k$ $(k \geq 1)$ be the sequence formed up to step k. Let j_{k+1} be the node furthest away in $A_2 \backslash A_1$ from j_k in the current network and let j_{k+1} play a best response where the only link that he/she has in A_2 is the link with j_k, *if such a best response exists*, and reiterate this step as far as such a best response exists until a wheel containing all nodes in $A_2 \backslash A_1$ and node j_1 is formed. Otherwise, step 2 must be recommenced by picking a node in $A_1 \cap A_m$ linked by a node in $A_1 \backslash A_m$ until a wheel containing all nodes in $A_m \backslash A_1$ and the chosen one in $A_1 \cap A_m$ is formed. It can be checked that now he formation of this wheel cannot be hindered by the non-existence of the desired best response. Now proceed as in the linear case (steps 3 and 4 apply unchanged) up to the completion of a wheel or a sequence of wheels (in this case contacting at isolated nodes at the intersection of two consecutive societies) including all nodes in all societies but one, say A_r. Note that now all nodes except possibly some in $A_r \backslash (A_{r+1} \cup A_{r-1})$ observe all nodes in N. Now proceed once more as in step 2, by picking a node in $A_{r-1} \cap A_r$ linked by a node in A_{r-1} and forming a wheel with all nodes in $A_r \backslash (A_{r+1} \cup A_{r-1})$ and the chosen one in $A_{r-1} \cap A_r$. Then, unless $A_{r-1} \cap A_r$ is a singleton, apply steps 3 ans 4 to merge this wheel with the one including all nodes in A_{r-1}. At this stage, some nodes at $A_r \cap A_{r+1}$ may support some unnecessary links with nodes in $A_r \backslash A_{r+1}$. Finally, let these players play best responses and delete these links, then a strict Nash $\mathcal{K}$-network consisting of either an all-encompassing wheel or a sequence of wheels containing all nodes is formed.

(iii) Finally, if the core of a societal cover $\mathcal{K}$ contains at least as many nodes as there are societies in $\mathcal{K}$, it is not difficult to adapt the proof, "expanding" an initial wheel containing all nodes in a society. A sufficient number of nodes within the core ensures that all such expansions are feasible and necessarily lead to an all-encompassing wheel. ■

[11] By contrast with the linear case, this is now not guaranteed.

[12] In the worst case, after the attempt in $m-1$ societies fails.

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