One-way flow network formation under constraints*

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Abstract

We study the effects of institutional constraints on stability and efficiency in the "one-way flow" model of network formation. In this model the information that flows through a link between two players runs only towards the player that initiates and supports the link, so in order for it to flow in both directions both players must pay whatever the unit cost of a directional link is. We assume that an exogenous "societal cover" consisting of a collection of possibly overlapping subsets covering the set of players specifies the social organization in different groups or "societies", so that a player may initiate links only with players that belong to at least one society that he/she also belongs to, thus restricting the feasible strategies and networks. In this setting, we examine the impact of such societal constraints on stable/efficient architectures and on dynamics.

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1 Introduction

In a seminal paper Bala and Goyal (2000) provide two benchmark non cooperative models of network formation. In both models links are formed unilaterally and the network allows information or other benefits to flow through it. In the "one-way flow" model the information flows through a link between two players only in the direction of the player that initiates and supports the link, so in order for it to flow in both directions both players must pay whatever the unit cost of a directional link is. In the "two-way flow" model the information flows through a link between two players in both directions irrespective of who pays for it.[1] In both settings, Bala and Goyal study Nash and strict Nash stability and provide a dynamic model, first assuming that information flows without friction or decay and then dropping this assumption. In both models the current network is assumed to be known by all players, who may unrestrictedly initiate links with any other players. These authors show that, when no friction exists, stability in the sense of Nash equilibrium is equivalent to minimal connectedness in either model, while in the stronger sense of *strict* Nash stability "wheels" are the only stable architectures in the one-way flow model and "center-sponsored stars" are the only stable architectures in the two-way flow model.

These benchmark models have been extended since then in different directions.[2] In Olaizola and Valenciano (2011) we argue that: *"Due to what is generically referred to here as "institutional constraints" (social, cultural, linguistic, geographical, economic, etc.), individuals may often see only "part of the world" and initiate links only within that part or a part of that part. Thus, it seems more realistic to assume that a set of possibly overlapping groups (family, tribe, clan, club, gender, age, linguistic community, nationality, professional association, department, etc., depending on the context) configures the social constraints within which individuals interact. More precisely, we assume that each individual may initiate links only within the groups he/she belongs to."* In that paper we address the same issues as Bala and Goyal (2000), but assume some institutional or social constraints. Namely, a "societal cover", consisting of a collection of groups of players called "societies" that covers the whole set of players, is exogenously given and it is assumed that each individual can only establish links with players with whom he/she shares membership of at least one society.[3] In Olaizola and Valenciano (2011) this study is conducted for the two-way flow model only. In the absence of decay, the strict Nash stable architectures are characterized and proved to exist for any societal cover and to be highly hierarchical in their organization: they form

[1] A third benchmark model is Jackson and Wolinsky's (1996), where the formation of a link between two players requires the agreement of both.

[2] There is a growing body of literature on these extensions that we decline to summarize here. Some recent books surveying part of this literature are Goyal (2007), Jackson (2008) and Vega-Redondo (2007). See also Jackson's (2010) survey.

[3] In Olaizola and Valenciano (2011) it is proved that the societal cover model provides the most general *symmetric* link-formation constraint that can be considered, i.e., any such constraint can be interpreted as associated with a societal cover.

oriented trees, perhaps "grafted", that collapse into Bala and Goyal's center-sponsored stars when the societal cover consists of a single society. Bala and Goyal's dynamic model is also applied to this extension and proved to converge at least in payoffs.

In this paper we address a similar study of link formation under institutional constraints *for the one-way flow model*, with a similar purpose: to see the impact of social constraints on Bala and Goyal's results. We obtain that this impact is stronger in the case of this model. We first prove equivalence between Nash stability and minimal connectedness. We then characterize strict Nash networks consistent with a societal cover as minimally connected networks formed by interconnected wheels where these interconnections satisfy a qualifying condition. This characterization is used to establish some features of strict Nash networks and, more importantly, to establish *their possible non existence*. This is a remarkable difference with the two-way flow model, where existence of strict Nash networks is guaranteed with and without constraints. Nevertheless, we prove their existence for some simple types of societal cover. Finally, we study Bala and Goyal's dynamics for the one-way flow model in this setting. We also show, unlike what happens in Bala and Goyal's setting, the *possibility of non convergence* even in payoffs. However, we prove convergence for the types of societal cover for which we have proved the existence of strict Nash networks, although in general their existence does not guarantee convergence to a strict Nash network. We also show that even strict Nash networks may be inefficient, and convergence to an efficient network is not guaranteed in general.

The paper closest to this one is Galeotti (2006), where equilibrium in the one-way flow model is studied in the presence of heterogeneity in costs and benefits.[4] Nevertheless, although our societal cover means a form of heterogeneity, Galeotti's model does not include, and is not included in, the one we study here.

The rest of the paper is organized as follows. In Section 2, the basic model is specified and the necessary notation and terminology are given. Section 3 studies stability and efficiency under institutional constraints. In Section 4, Bala and Goyal's dynamic model is applied to this setting. Finally, Section 5 summarizes the main conclusions.

2 The model: one-way flow under constraints

We describe formally the ingredients of the model by recalling some definitions from Bala and Goyal (2000) and Olaizola and Valenciano (2011).

Let $N = \{1, 2, .., n\}$ denote the set of *players*. Each player may choose other players with whom to initiate and support *links*. By $g_{ij} \in \{0, 1\}$ we denote the existence ($g_{ij} = 1$) or not ($g_{ij} = 0$) of a link connecting i and j initiated by i, and when such a link exists we refer to it as "link $\overleftarrow{ij}$" and write $\overleftarrow{ij} \in g$ meaning $g_{ij} = 1$. Vector

[4] Other papers dealing with heterogeneity in the one-way flow model are Billand et al. (2008), Derks and Tennekes (2009), and Derks et al. (2009).

$g_i = (g_{ij})_{j \in N \backslash i} \in \{0,1\}^{N \backslash i}$ specifies[5] the set of links supported by i and is referred to as an (unrestricted) *strategy* of player i. $G_i := \{0,1\}^{N \backslash i}$ denotes the set of i's (unrestricted) strategies and $G_N = G_1 \times G_2 \times .. \times G_n$ the set of (unrestricted) strategy profiles. An unrestricted strategy profile $g \in G_N$ univocally determines a directed *N-network* (N, Γ_g), where

$$\Gamma_g := \{(i,j) \in N \times N : g_{ij} = 1\},$$

which we identify with g and refer to as network g. If $M \subseteq N$ we denote by $g\mid_M$ the *M-network* $(M, \Gamma_{g|_M})$ with

$$\Gamma_{g|_M} := \{(i,j) \in M \times M : g_{ij} = 1\},$$

which we refer to as the *M-subnetwork* of g. As N is usually clear from the context, we generally write just "network" instead "N-network".

We assume that an exogenous "societal cover" consisting of a set of possibly overlapping "societies" imposes a social constraint: each player in N can initiate links only with players with whom he/she shares membership of at least one society. Formally, we have the following

Definition 1 *(Olaizola and Valenciano, 2011) A "societal cover" of N is a collection of subsets of N (called "societies"), $\mathcal{K} \subseteq 2^N$, such that: (i) $\bigcup_{A \in \mathcal{K}} A = N$, and (ii) for all $A, B \in \mathcal{K}$ $(A \neq B)$, $A \nsubseteq B$.*

Thus, every player belongs to at least one society and no society contains any other.

We denote by $\mathcal{K}_i \subseteq \mathcal{K}$ the *affiliation* of player i or set of *societies* to which i belongs, and by $N(\mathcal{K}_i) \subseteq N$ player i's *reach,* i.e., the set of players that i may directly access, that is:

$$\mathcal{K}_i := \{A \in \mathcal{K} : i \in A\}$$

and

$$N(\mathcal{K}_i) := \bigcup_{A \in \mathcal{K}_i} A.$$

A *component $\mathcal{C}$ of a societal cover $\mathcal{K}$* is a subset $\mathcal{C} \subseteq \mathcal{K}$ such that *(i)* for all $A, B \in \mathcal{C}$ there exist $A_1, .., A_k \in \mathcal{K}$ s.t. $A_1 = A$ and $B = A_k$, and $A_i \cap A_{i+1} \neq \varnothing$ for $i = 1, .., k-1$, and *(ii)* for all $B \in \mathcal{K} \backslash \mathcal{C}$, $B \cap (\cup_{A \in \mathcal{C}} A) = \varnothing$. The subset $\cup_{A \in \mathcal{C}} A$ of N covered by a component $\mathcal{C}$ is denoted by $N(\mathcal{C})$. A societal cover is *connected* if it has a unique component.

The following definition constrains the structure of a network so as to be consistent with a given societal cover of N.

[5] We always drop the brackets "{..}" in expressions such as $N \backslash \{i\}$.

Definition 2 *(Olaizola and Valenciano, 2011) A network g is consistent with a societal cover $\mathcal{K}$ (or is a $\mathcal{K}$-network) if for every link $g_{ij} = 1$ there exists an $A \in \mathcal{K}$ s.t. $i, j \in A$ (i.e., $\mathcal{K}_i \cap \mathcal{K}_j \neq \varnothing$).*

A vector $g_i = (g_{ij})_{j \in N(\mathcal{K}_i) \setminus i} \in \{0,1\}^{N(\mathcal{K}_i)\setminus i}$ specifies a set of $\mathcal{K}$-*feasible* links initiated by i and is referred to as a $\mathcal{K}$-*admissible strategy* of player i. $G_i(\mathcal{K}) := \{0,1\}^{N(\mathcal{K}_i)\setminus i}$ denotes the set of i's $\mathcal{K}$-admissible strategies and $G_{\mathcal{K}} = G_1(\mathcal{K}) \times G_2(\mathcal{K}) \times .. \times G_n(\mathcal{K})$ the set of $\mathcal{K}$-admissible strategy profiles. A $\mathcal{K}$-admissible strategy profile g determines a $\mathcal{K}$-network that is identified with g.

We say that there is a *path of length k from j to i* in g if there exist $k+1$ players $j_0, j_1, .., j_k$, s.t. $i = j_0$, $j = j_k$, and for all $l = 1, .., k$, $g_{j_{l-1}j_l} = 1$. When such a path exists we write $j \xrightarrow{g} i$. Note that the *direction* of a path is crucial: a path from j to i allows information or benefits to travel from j to i, but *not* from i to j. The set of players with whom i supports a link is denoted by $N^d(i; g)$, and the set of players connected with i by a path (union $\{i\}$) by $N(i; g)$, and their cardinalities by $\mu_i^d(g) := \#N^d(i; g)$ and $\mu_i(g) := \#N(i; g)$.

A *component* of a network g is a subnetwork $g \mid_C$, where $C \subseteq N$, such that for any two players j, i in C, $j \xrightarrow{g} i$, and no set strictly containing C meets this condition. We say that g is *connected* if g has a unique component. A component of a network is *minimal* if for all i, j s.t. $g_{ij} = 1$, the number of components of g is smaller than the number of components of $g - \overleftarrow{ij}$, where $g - \overleftarrow{ij}$ is the network that results from replacing $g_{ij} = 1$ by $g_{ij} = 0$ in g. A network is *minimally connected* if it is connected and minimal.

We denote by g_{-i} the network where all links supported by i in g are deleted, and by (g_{-i}, g_i') the strategy profile and network that results from replacing g_i by g_i' in g.

It is assumed that each player has a valuable and particular type of information and a link $\overleftarrow{ij}$ allows such information to flow from j to i, without friction or decay, so that each player i receives the information from all players with whom he/she is connected by a path, i.e., all j such that $j \xrightarrow{g} i$. Let $v_{ij} > 0$ be the payoff that player i derives from connecting directly (by a link) or indirectly (by a path) with player j, and $c_{ij} > 0$ the cost for player i of initiating a link with j. Thus, the payoff of player i in g is

$$\Pi_i(g) = \sum_{j \in N(i;g)} v_{ij} - \sum_{j \in N^d(i;g)} c_{ij}.$$

We assume costs and benefits to be homogeneous across players (i.e., $v_{ij} = v$ and $c_{ij} = c$, for all i, j).[6] We also assume $v > c$, so that connecting with *new* players (i.e.,

[6] In fact, the constraint imposed by a societal cover means a form of heterogeneity that could be formulated in terms of payoffs. For this assume that each player i has a cost

$$c_{ij} \begin{cases} = c, & \text{if } j \in N(\mathcal{K}_i) \\ = M, & \text{otherwise;} \end{cases}$$

those with whom one player is not connected by a path) is always profitable. In short, we assume:

$$\Pi_i(g) = v\mu_i(g) - c\mu_i^d(g) \quad \text{and} \quad v > c. \tag{1}$$

A $\mathcal{K}$-network is *efficient* if it maximizes the aggregate payoff under the constraint of $\mathcal{K}$-feasible payoffs, that is, those that can be obtained by means of $\mathcal{K}$-networks.

3 Stability and efficiency

The following definitions are natural extensions of the notions of Nash stability and strict Nash stability, following Bala and Goyal (2000), for a network in a scenario where payoffs are given by (1) and: (i) a societal cover $\mathcal{K}$ allows only for links connecting individuals that have at least one society in common, and (ii) all players in the same component $\mathcal{C}$ of $\mathcal{K}$, i.e., in $N(\mathcal{C})$, know the part of the current network connecting individuals of $N(\mathcal{C})$. Condition (ii) can be justified by assuming that information about which is the current network propagates between overlapping societies.[7] Note that this scenario yields the unconstrained environment of Bala and Goyal (2000) for the particular case of the simplest societal cover: $\mathcal{K} = \{N\}$.

Definition 3 *A Nash $\mathcal{K}$-network is a $\mathcal{K}$-network g that is stable under $\mathcal{K}$-admissible strategies, that is, for all $i \in N$:*

$$\Pi_i(g) \geq \Pi_i(g_{-i}, g_i') \quad \textit{for all } g_i' \in G_i(\mathcal{K}). \tag{2}$$

When (2) holds, we say that g_i is a *best (admissible) response* of i to g_{-i}. The stability notion can be refined in the strict sense:

Definition 4 *A strict Nash $\mathcal{K}$-network is a Nash $\mathcal{K}$-network g such that for all $i \in N$:*

$$\Pi_i(g) > \Pi_i(g_{-i}, g_i') \quad \textit{for all } g_i' \in G_i(\mathcal{K}) \ (g_i' \neq g_i). \tag{3}$$

Thus, (3) means that in a strict Nash $\mathcal{K}$-network every player is playing his/her *unique* best (admissible) response to those played by the others. Note that for $\mathcal{K} = \{N\}$ a (strict) Nash $\mathcal{K}$-network is a (strict) Nash network in the standard setting.

Note that the set of players in each component of the societal cover form a separate world: no link with players in other components is possible and no information about them reaches them. In particular the following straightforward result follows easily.

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where M is a sufficiently large number, and $v_{ij} = v$ for all j. Note the difference with the other forms of heterogeneity that, as far as we know, have been considered in the literature.

[7]As a set, each society can be conceived as a *complete* undirected network connecting all its members that allows for a certain level of information to flow, including which the current $\mathcal{K}$-network is, but *not* the information that flows through it. The underlying N-graph consisting of all these societal complete networks as subgraphs is the underlying societal network which prescribes the feasible links. There is no conflict with the interpretation of the model if we assume that the $\mathcal{K}$-network we consider now, with associated costs and benefits, allows for the flow of a particular type of information that cannot flow through the underlying network imposed by the societal cover.

Proposition 1 *A $\mathcal{K}$-network g is a Nash (strict Nash) $\mathcal{K}$-network if and only if $g\mid_{N(\mathcal{C})}$ is a Nash (strict Nash) $\mathcal{C}$-network for each component $\mathcal{C}$ of $\mathcal{K}$.*

Remark: In view of Proposition 1 and the irrelevance of societies consisting of a single individual, in what follows *our attention is constrained to connected societal covers.*

The following result, characterizing Nash $\mathcal{K}$-networks, extends Bala and Goyal's result to this setting.

Theorem 1 *Given a connected societal cover $\mathcal{K}$ of N, a $\mathcal{K}$-network g is a Nash $\mathcal{K}$-network if and only if it is minimally connected.*

Proof. *Necessity* ($\Rightarrow$): Let $\mathcal{K}$ be a connected societal cover of N, and g a Nash $\mathcal{K}$-network. Assume g is not connected. Then there exist two players $i, j \in N$ such that there is no path from j to i in g. As cover $\mathcal{K}$ is connected, a finite sequence of players $i_1, .., i_m$ exists, such that $i_1 = i$, $i_m = j$ and for each $k = 1, .., m-1$, there is an $A \in \mathcal{K}$ s.t. $i_k, i_{k+1} \in A$. Then for at least two consecutive players among these m players, say i_k and i_{k+1}, there is no path in g from i_{k+1} to i_k. But then it is feasible and profitable for player i_{k+1} to initiate a link with i_k, and this contradicts that g is a Nash $\mathcal{K}$-network. Thus g must be connected. Finally, if g were connected but not minimal, there would be some superfluous link that could be eliminated and that would benefit the player that eliminated it, and consequently g would not be a Nash $\mathcal{K}$-network.

Sufficiency ($\Leftarrow$): Reciprocally, we prove that if g is *not* a Nash $\mathcal{K}$-network then g is *not* minimally connected. Assume that there is some player i s.t. $\Pi_i(g') > \Pi_i(g)$, where $g' = (g_{-i}, g'_i)$ for some strategy $g'_i \in G_i(\mathcal{K})$, and assume that g is connected (otherwise we are done). Without loss of generality we assume that g'_i is a best response to g_{-i}, which implies $N(i; g') = N(i; g) = N$. Since g is connected, player i must support at least one link in g. If i supports only one link in g, then i cannot improve his/her payoff. Then assume that i supports at least two links in g. As g is connected and g'_i is a best response, $\Pi_i(g') > \Pi_i(g)$ means that i supports a *smaller* number of links in g' than in g. If i improves his/her payoff by just severing some links in g, then g would not be minimally connected and we would be done. Then, suppose that i deletes some links in g and replaces them by a *smaller* number of links. If g were minimally connected, each deleted link is necessary to connect i with at least one player in g, therefore each new link (i.e., every link with players in $N^d(i; g') \backslash N^d(i; g)$) must be necessary to connect i with at least one of the disconnected ones (i.e., those in $N^d(i; g) \backslash N^d(i; g')$), otherwise g'_i would not be a best response. Note that for any two of the deleted links, say, $\overleftarrow{ik}$ and $\overleftarrow{ik'}$, as g is connected, k and k' are connected in g, but observe that, unless $\overleftarrow{ik}$ and $\overleftarrow{ik'}$ are superfluous in g, they are necessary for the connections $k \overset{g}{\to} k'$ and $k' \overset{g}{\to} k$ respectively, forming the first link in the path, that is, we have $k \overset{g}{\to} i \overset{g}{\to} k'$ and $k' \overset{g}{\to} i \overset{g}{\to} k$, where in both cases the first path consists of a single link. Therefore, as the number of new links in g' is smaller than the number of severed ones, at least one of the new links, say, $\overleftarrow{ij} \in g'$, must be necessary to connect i with *two* of the disconnected players, say,

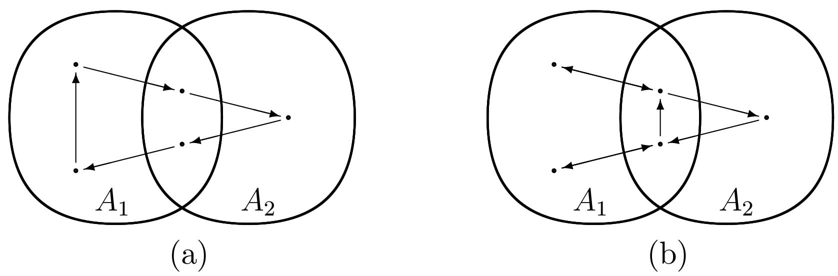

Figure 1: Efficient and inefficient Nash $\mathcal{K}$-networks

k and k', in g'. As $\overleftarrow{ij} \in g'$ connects i with k and k' in g', we have $k \overset{g'}{\to} j$ and $k' \overset{g'}{\to} j$, but these paths do not contain deleted links nor any of the new ones. For this note that if a path $k \overset{g'}{\to} j$ contains one of the new links, say, $\overleftarrow{ij'} \in g'$, $\overleftarrow{ij} \notin g$, then $\overleftarrow{ij}$ would not be necessary to connect i with k in g'. Thus we have $k \overset{g}{\to} j$ and, by an entirely similar reasoning, $k' \overset{g}{\to} j$ not containing deleted links. Finally, there must exist $j \overset{g}{\to} i$ not containing any deleted link, because if it contained a deleted link, say $\overleftarrow{ik''}$, we have a path $k \overset{g}{\to} i$ of the form $k \overset{g}{\to} j \overset{g}{\to} k'' \overset{g}{\to} i$ which would make superfluous $\overleftarrow{ik}$ and $\overleftarrow{ik'}$ in g. Thus we have proved the existence of paths in g:

$$k \overset{g}{\to} j \overset{g}{\to} i \quad \text{and} \quad k' \overset{g}{\to} j \overset{g}{\to} i$$

not using deleted links, thus making superfluous $\overleftarrow{ik}$ and $\overleftarrow{ik'}$ in g. In other words, assuming that g is minimally connected and not Nash leads to contradiction. ■

As minimally connected $\mathcal{K}$-networks always exist, an immediate consequence is the following.

Corollary 1 *For any connected societal cover $\mathcal{K}$ of N, a Nash $\mathcal{K}$-network exists.*

Remark: Minimal connectedness is a necessary condition for efficiency, but, by contrast with the two-way flow model of Bala and Goyal (2000) and its extension in Olaizola and Valenciano (2011), in the one-way flow model a $\mathcal{K}$-network may be minimally connected but not efficient. In Figure 1 two minimally connected $\mathcal{K}$-networks are represented[8]: (a) is an efficient Nash $\mathcal{K}$-network, while (b) is a Nash $\mathcal{K}$-network that is not efficient. A similar situation occurs within Bala and Goyal's one-society setting, but in that case efficiency is at least guaranteed for the unique architecture of *strict* Nash networks, that is, the wheel. But, as we presently show, this is not the case in general in the more general setting considered here.

[8]As in all figures, players are represented by dots (without labels unless convenient), and links by arrows between them with the convention that the player at the tip of the arrow supports it.

We now concentrate on *strict* Nash $\mathcal{K}$-networks. Bala and Goyal (2000) prove that in their single-society setting wheels are the only strict Nash architectures. In their setting a wheel is a sequence of players that *includes all* and where each player supports a (unique) link with the next one in the sequence and the last one with the first, or, equivalently, a minimally connected network where each player supports exactly one link. In the context of $\mathcal{K}$-networks an all-encompassing wheel may not even be feasible for certain covers, but, as we show below, wheels are also important in connection with strict Nash $\mathcal{K}$-networks. This motivates the following definition.

Definition 5 *Let g be an N-network and $M \subseteq N$ ($\#M \geq 2$) a subset of players, if for a certain permutation of M, $i_1, i_2, .., i_m$, we have $g_{i_k i_{k+1}} = 1$ ($k = 1, .., m-1$) and $g_{m1} = 1$, and no other link exist between any two of players in M, then $g\mid_M$ is called a wheel.*

In this case we say that the players in M form a wheel in g. In fact, this is equivalent to say that $g\mid_M$ is a minimally connected M-network where each player supports exactly one link. Note that according to this definition: (i) a wheel does *not* necessarily connect all players in N; (ii) no link connects any pair of players in the same wheel unless they are consecutive on it; (iii) a player in a wheel can link, or be linked by, other players different from those in the wheel. When $M = N$ we say that g is an *all-encompassing wheel*.

Re-stated in terms of the current setting, notation and terminology, and adapted to it, Bala and Goyal (2000) establish the following result: *when $\mathcal{K} = \{N\}$, i.e., the cover consists of a single society, the only strict Nash $\mathcal{K}$-networks are all-encompassing wheels.*[9]

As we show below, the societal cover diversifies the strict Nash networks. A variety of structures of interconnected wheels emerges as possible strict Nash $\mathcal{K}$-networks depending on the structure of the societal cover; moreover, in general, several architectures appear as strict Nash for a given societal cover. But let us first see that Nash networks consist of interconnected wheels.

Proposition 2 *In a Nash $\mathcal{K}$-network every player belongs to at least one wheel in g.*

Proof. Let g be a Nash $\mathcal{K}$-network. By Theorem 1, g is minimally connected. Let i and j be two players. As g is connected, there exist paths $i \xrightarrow{g} j$ and $j \xrightarrow{g} i$. Thus there is a cycle, i.e., a sequence of players $i_1, .., i_m$ such that $i_1 = i_m = i$ and $g_{i_l i_{l+1}} = 1$. We can assume that i only appears at the ends of this sequence (if i appears somewhere else in this sequence just take a subsequence where this condition holds). Now consider this cycle if no player appears twice on it, otherwise delete the subcycles until no player appears twice. Then the players in the cycle so obtained form a wheel in g because: (i) as g is minimally connected, no two players in the cycle are mutually linked unless

[9] Given their weaker assumptions on payoffs, the empty network may also be strict Nash in their setting.

the cycle contains only two players, and (ii) by the same reason, no link connects any pair of players in the cycle unless they are consecutive on it. ■

It can easily be shown by a counterexample that not every connected $\mathcal{K}$-network formed by wheels is minimally connected nor therefore Nash.

Our next goal is to identify and characterize strict Nash $\mathcal{K}$-networks. As a first result, we have a characterization that will allow us to establish some salient features of strict Nash $\mathcal{K}$-networks and prove their *existence or non-existence under certain conditions*. Namely, we have the following characterization: strict Nash $\mathcal{K}$-networks are minimally connected $\mathcal{K}$-networks (therefore consisting of wheels) which satisfy a qualifying condition that restricts the way in which those wheels can interconnect.

Theorem 2 *A network g is a strict Nash $\mathcal{K}$-network if and only if the following two conditions hold: (i) g is a minimally connected $\mathcal{K}$-network, and (ii) for all $\overleftarrow{ij} \in g$ and all $k \neq j$ s.t. $j \in N(k; g - \overleftarrow{ij})$, $k \notin N(\mathcal{K}_i)$.*

Proof. *Necessity* ($\Rightarrow$): Let g be a strict Nash $\mathcal{K}$-network. By Theorem 1, g is minimally connected. Now assume that for some i it holds $g_{ij} = 1$ and for some $k \neq j$ s.t. $j \in N(k; g - \overleftarrow{ij})$, it is $k \in N(\mathcal{K}_i)$. Then i can replace the link with j by a link with k and keep his/her payoff.

Sufficiency ($\Leftarrow$): Let g be a $\mathcal{K}$-network satisfying conditions (i) and (ii). By Theorem 1, g is a Nash $\mathcal{K}$-network. Assume that $g_i' \in G_i(\mathcal{K})$ is a best response of i to g_{-i}, and $g_i' \neq g_i$. Then i cannot support more links in $g' = (g_{-i}, g_i')$ than in g, otherwise i's payoff would be smaller in g' than in g. But i cannot support less and keep the payoff, otherwise g would not be a Nash $\mathcal{K}$-network. Thus, a best response g_i' must consist of replacing a number of links by the same number of new links. As g is connected, i must support at least one link in g. If i supports in g only one link, by condition (ii) i cannot replace this link by any other without a loss. Assume then that i deletes at least two links and replaces them by an equal number of new ones. As g is minimally connected, by a similar argument to that in Theorem 1, for any pair k, k' of disconnected players (i.e., $k, k' \in N^d(i; g) \backslash N^d(i; g')$), $k \overset{g}{\to} k'$ and $k' \overset{g}{\to} k$ contain necessarily $\overleftarrow{ik}$ and $\overleftarrow{ik'}$ respectively. Thus each new link in g_i' must be critical to receive one of the disconnected ones. Let $\overleftarrow{ij} \in g_i'$ be critical to receive $k \in N^d(i; g) \backslash N^d(i; g')$, and $\overleftarrow{ij'} \in g_i'$ be critical to receive k'. Then there must exist a path $k \overset{g'}{\to} j$, but this path cannot contain any of the new links: if it contained, say $\overleftarrow{ij'}$, then $\overleftarrow{ij}$ would not be critical for i to receive k in g'. Therefore such path is a path $k \overset{g}{\to} j$ preserved in g' and consequently not using any deleted link. Therefore $j \in N(j; g - \overleftarrow{ik})$, but then condition (ii) implies $j \notin N(\mathcal{K}_i)$, that is, $\overleftarrow{ij}$ is not feasible. ■

As a consequence the following is obtained.

Corollary 2 *Any all-encompassing wheel that is a $\mathcal{K}$-network is a strict Nash $\mathcal{K}$-*

network. In particular, when $\mathcal{K} = \{N\}$ the only strict Nash $\mathcal{K}$-networks are all-encompassing wheels.

Proof. First note that if an all-encompassing wheel is feasible for $\mathcal{K}$, it is evidently minimally connected and condition (ii) of Theorem 2 holds trivially, given that for no link $\overleftarrow{ij} \in g$ there is $k \neq j$ s.t. $j \in N(k; g - \overleftarrow{ij})$. And when $\mathcal{K} = \{N\}$, as any player is within reach of any other node, the only way in which condition (ii) of Theorem 2 may hold for a minimally connected $\mathcal{K}$-network is when for all $\overleftarrow{ij} \in g$ there is no $k \neq j$ s.t. $j \in N(k; g - \overleftarrow{ij})$, which entails that every player supports a single link, which yields an all-encompassing wheel. ■

Now based on the characterization provided by Theorem 2, we establish some salient features of the architecture of strict Nash $\mathcal{K}$-networks. The first one gives a necessary condition for the contact and existence of *two* wheels in strict Nash $\mathcal{K}$-networks: the existence of a "hinge-player", that is, a unique player in the intersection of the reaches of another two.

Corollary 3 *In a strict Nash $\mathcal{K}$-network g if a player i is linked by two different players j and k, i.e. $g_{ji} = g_{ki} = 1$ with $j \neq k$, then i is the unique player in the intersection of j's reach and k's reach, that is, $N(\mathcal{K}_j) \cap N(\mathcal{K}_k) = \{i\}$.*

Proof. Let i, j, k be three players in a strict Nash $\mathcal{K}$-network g s.t. $g_{ji} = g_{ki} = 1$. Assume there is a player $i' \neq i$, which is within j's reach. As g is minimally connected, there is a path $i \overset{g}{\to} i'$. As $i'(\neq i)$ is within j's reach, any such path must contain link $\overleftarrow{ji}$, otherwise condition (ii) of Theorem 2 would be violated. Thus, there is a path $i \overset{g}{\to} i'$ not containing link $\overleftarrow{ki}$, but then condition (ii) of Theorem 2 implies that $i'(\neq k)$ is *not* within k's reach. ■

As a corollary, there are two conclusions: a sufficient condition under which the only possible architecture of a strict Nash $\mathcal{K}$-network is the all-encompassing wheel; and a fact: within a society information may "converge" at a player but never "diverge" in a strict Nash $\mathcal{K}$-network. Formally, we have the following result.

Corollary 4 *If g is a strict Nash $\mathcal{K}$-network, then: (i) if no two societies in cover $\mathcal{K}$ intersect in a singleton, g is necessarily an all-encompassing wheel; (ii) for any three players i, j, k within the same society, $g_{ij} = g_{ik} = 1$ may hold, but $g_{ji} = g_{ki} = 1$ is not possible.*

Proof. (i) By Proposition 2, a strict Nash $\mathcal{K}$-network is formed by interconnected wheels, and the existence of two or more wheels means necessarily that some player is linked by two different players. By Corollary 3, such a player must be the unique player at the intersection of the reaches of these two, or, equivalently, any two societies containing this player and either of the other two must share only that player. Now if no two societies in cover $\mathcal{K}$ intersect in a singleton this is not possible, thus only an all-encompassing wheel can be a strict Nash $\mathcal{K}$-network.

(ii) Let i, j, k be three players in the same society. By Corollary 3, no three players for which $g_{ji} = g_{ki} = 1$ in a strict Nash $\mathcal{K}$-network g can belong to the same society. But the following example shows that $g_{ij} = g_{ik} = 1$ is possible. Assume $N = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{K} = \{\{1, 2, 3\}, \{1, 4\}, \{2, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}\}$, $g_{12} = g_{13} = g_{41} = g_{54} = g_{64} = g_{25} = g_{36} = 1$, otherwise $g_{ij} = 0$. Thus we have $g_{12} = g_{13} = 1$ and $1, 2, 3$ belong to the same society, and it is easy to check that g is a strict Nash $\mathcal{K}$-network. ■

In Olaizola and Valenciano (2011) the characterization of strict Nash $\mathcal{K}$-networks for the two-way flow model under the constraints imposed by a societal cover allows an easy constructive proof of the existence of such networks. In contrast with this result, the question of existence in the one-way flow model is answered *negatively* by the preceding consequences of the characterization and the following example.

Example 1: Let $N = \{1, 2, 3, 4, 5\}$ and let the cover $\mathcal{K} = \{\{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}$. The *non existence of a strict Nash $\mathcal{K}$-network* for this cover can be derived independently from Corollary 3 and from Corollary 4. Given that, as the reader can easily check, no all-encompassing wheel is feasible for this cover, non existence of strict Nash networks follows from Corollary 4-i. But this conclusion also follows directly from Corollary 3: in order to be connected, players $1, 2$ and 3 must link to either 4 or 5, but then necessarily two of them link to the same player and whoever they are their reaches share *two* players.

This non-existence result raises the question of conditions for a cover $\mathcal{K}$ under which a strict Nash $\mathcal{K}$-network does exist. It seems too hard obtaining a sufficiently general sufficient condition, and each particular cover is a case-study. To give a hint of how to address this issue, we now describe some simple "regular" covers for which the existence of strict Nash $\mathcal{K}$-networks can be easily be established.

Let us call *linear* societal covers those of the form $\mathcal{K} = \{A_i\}_{i=1,2,..,m}$, where for all $i = 1, 2, .., m-1$, $A_i \cap A_{i+1} \neq \varnothing$, and in all other cases two societies do not intersect.

Let us call *covers in wheel* those of the form $\mathcal{K} = \{A_i\}_{i=1,2,..,m}$ s.t. for all $i = 1, 2, .., m-1$, $A_i \cap A_{i+1} \neq \varnothing$, and $A_1 \cap A_m \neq \varnothing$, and in all other cases two societies do not intersect.

The *societal core* (Olaizola and Valenciano, 2011) of a cover $\mathcal{K}$ is the set of players that belong to all societies, i.e., $core(\mathcal{K}) := \cap_{A \in \mathcal{K}} A$. Then we have the following.

Proposition 3 *Let $\mathcal{K}$ be a societal cover, then in the following cases a strict Nash network does exist: (i) $\mathcal{K}$ is linear; (ii) $\mathcal{K}$ is a cover in wheel; (iii) $\mathcal{K}$ has a nonempty core that contains at least as many players as the number of societies in $\mathcal{K}$,*

Proof. (i) First consider the case where for all $i = 1, 2, .., m-1$, $\#(A_i \cap A_{i+1}) > 1$. In this case the all-encompassing wheel is $\mathcal{K}$-feasible and therefore strict Nash $\mathcal{K}$-networks exist (see Fig. 2 (a)). Moreover, in view of Corollary 4-i, this is the only possible architecture of a strict Nash $\mathcal{K}$-network. Now consider the case where for some i, $\#(A_i \cap A_{i+1}) = 1$. In this case the all-encompassing wheel is not feasible, but a strict Nash $\mathcal{K}$-network can be constructed by using wheels to connect players in any subsequence of consecutive societies whose intersection contains more than one player. Then these wheels interconnect by means of the players at those intersections that contain a single player, thus forming a strict Nash $\mathcal{K}$-network (see Fig. 2 (b)).

(ii) Consider a cover *in wheel*. For such covers the all-encompassing wheel is $\mathcal{K}$-feasible and therefore strict Nash. Just pick m players $i_1, .., i_m$, with $i_j \in A_j \cap A_{j+1}$, for $i = 1, 2, .., m-1$, and $i_m \in A_1 \cap A_m$, then starting at i_1, connect all unconnected player within society A_j forming a path $i_j \to i_{j+1}$ till a wheel is completed (see Fig. 2 (c)). Thus, by Corollary 4-i, when all intersections contain at least two players the all-encompassing wheel is the only architecture of a strict Nash $\mathcal{K}$-network; while when some intersection contains a single player, architectures other than the all-encompassing wheel are also feasible for a strict Nash $\mathcal{K}$-network (see Fig. 2 (d)).

(iii) When the core of a cover $\mathcal{K}$ is not empty and contains at least as many players as the number of societies in $\mathcal{K}$, then the all-encompassing wheel is $\mathcal{K}$-feasible and therefore strict Nash. Just pick as many players in the core as there are societies in the cover, say m, $\{i_1, .., i_m\} \subset core(\mathcal{K})$. Starting at i_1 connect all unconnected players within $A_j \backslash \{i_{j+2}, .., i_m\}$ forming a path $i_j \to i_{j+1}$ till a wheel is completed (see Fig. 2 (e)). Observe that this condition does not hold in Example 1, where no strict Nash $\mathcal{K}$-network exists, but note also that this sufficient condition is not necessary (see, e.g., Fig. 2 (f)). ■

Thus, the existence of hinge-players, i.e., unique players in the intersection of two societies, is a necessary condition for the existence of multiple wheels in a strict Nash $\mathcal{K}$-network, but their existence does not entail this multiplicity to be necessary (see, e.g., Fig. 2 (c) and (d)). A *sufficient* condition for multiplicity of wheels to be necessary in a strict Nash $\mathcal{K}$-network is the following. A *strong* hinge-player for a cover $\mathcal{K}$ of N is a player $i \in N$ s.t. $\mathcal{K} \mid_{N\backslash i}$ is a not connected cover of $N \backslash i$, where $\mathcal{K} \mid_{N\backslash i} = \{A \backslash i : A \in \mathcal{K}\}$.[10] The existence of such players imposes multiplicity of wheels in any strict Nash $\mathcal{K}$-network, but this condition is *not* necessary for the existence of strict Nash $\mathcal{K}$-networks with multiple wheels: in Fig. 2 (d) the hinge-player connecting the two wheels is not a *strong* hinge-player.

Remarks about efficiency: In view of the results of this section Nash and strict Nash $\mathcal{K}$-networks may be inefficient. Of course, when the all-encompassing wheel is feasible it is the unique efficient architecture. For instance, for a societal cover in wheel only the all-encompassing wheel is efficient, but other strict Nash architectures exist when there are hinge-players (e.g., Fig. 2 (d)). In this case, the smaller the number of wheels composing a strict Nash $\mathcal{K}$-network the "more efficient". In linear societal covers hinge-players impose multiple wheels, but in this case there is a unique strict Nash architecture and consequently all strict Nash $\mathcal{K}$-networks are efficient (e.g.,

[10] In terms of graph theory, a strong hinge-player is an articulation node of the underlying constraining network (see Footnote 7), that is, a node i such that the underlying constraining network restricted to $N \backslash i$ is not connected. This notion was suggested to us by a comment of a referee.

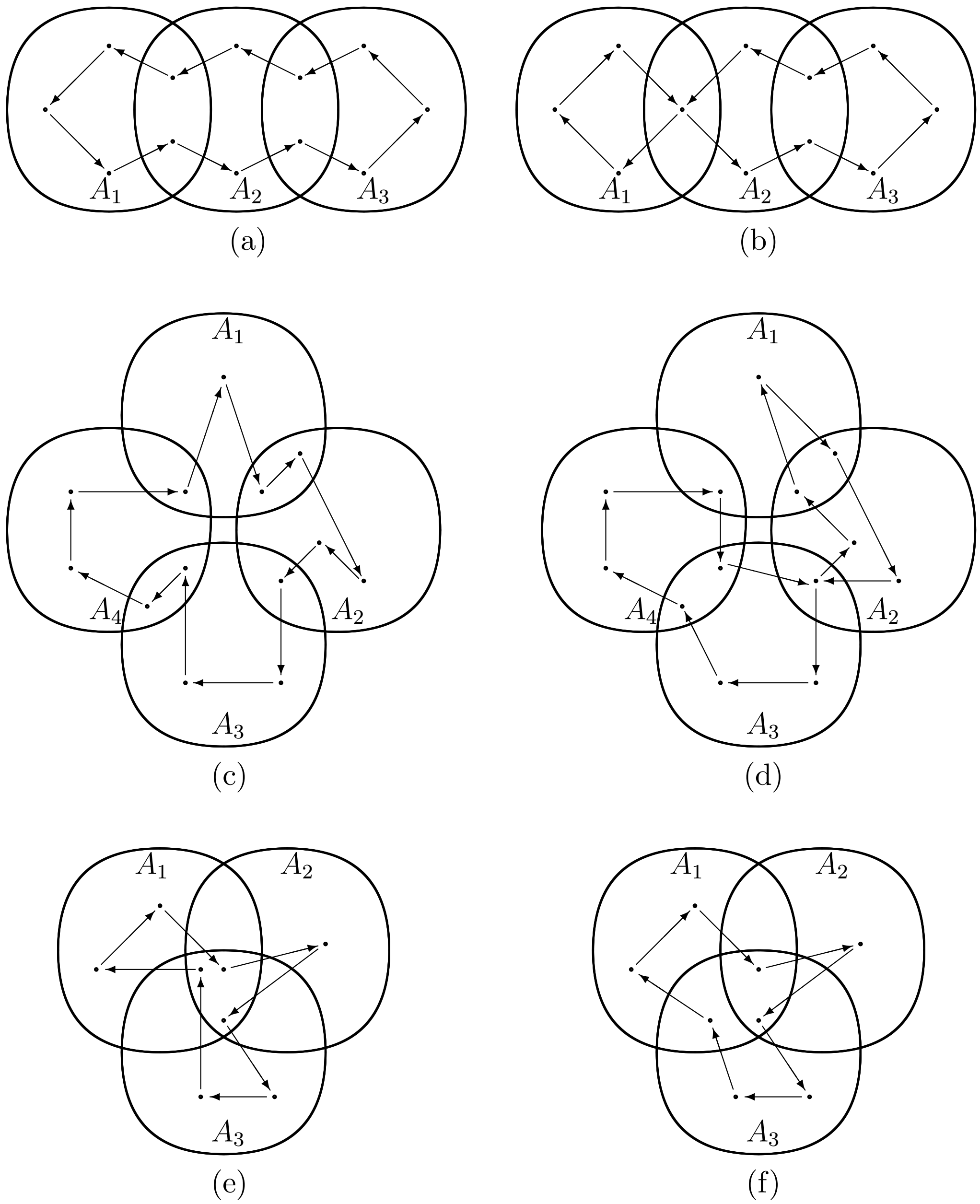

Figure 2: Strict Nash $\mathcal{K}$-networks

Fig. 2 (b)). In the case of a cover with a core containing a sufficiently large number of players the all-encompassing wheel is the unique, and consequently efficient, strict Nash architecture.[11]

4 Dynamics

We now apply Bala and Goyal's (2000) dynamic model in this setting. Namely, starting from any initial $\mathcal{K}$-network g, in each period each player i independently with a positive probability responds with a $\mathcal{K}$-admissible best response to g_{-i} (this includes any strategy that yields the same payoff to i as the current one when no strategy can improve i's payoff), or randomizes across them when there is more than one. Otherwise, player i exhibits *inertia*, i.e., keeps his/her links unchanged. Note that at each period several players may change strategy simultaneously. In this way, a Markov chain on the state space of all $\mathcal{K}$-networks is defined. Bala and Goyal prove that in their setting, i.e., for $\mathcal{K} = \{N\}$, starting from any network, the dynamic process converges to a strict Nash network with probability 1. In other words, the only absorbing sets[12] are singletons consisting of wheels.

Olaizola and Valenciano (2011) shows how Bala and Goyal's dynamic model for the two-way flow model may fail to converge to a strict Nash $\mathcal{K}$-network when a societal cover constrains link-formation. This possibility leads to the introduction of the notion of *quasi* strict Nash $\mathcal{K}$-networks, constituents of the absorbing sets for such dynamics, whose existence is proved. These *quasi* strict Nash $\mathcal{K}$-networks are just minimally connected $\mathcal{K}$-networks which are "miscoordination-proof", i.e., such that they cannot be disconnected by best response dynamics, and such that when the dynamic reaches one of them the payoffs remain stable for all players in spite of everlasting oscillations. The following example shows such a situation for the one-way flow model:

Example 2: Let $N = \{1,2,3,4,5,6\}$ and let $\mathcal{K}$ be the cover $\mathcal{K} = \{\{1,2\}, \{1,3\}, \{3,5,6\}, \{2,3,4,6\}\}$. Consider the $\mathcal{K}$-network g such that $g_{12} = g_{21} = g_{26} = g_{65} = g_{53} = g_{34} = g_{42} = 1$, and $g_{ij} = 0$ otherwise. Network g is represented in Figure 3. Given the affiliation of player 1, his/her only best response consists of replacing link $\overleftarrow{12}$ by $\overleftarrow{13}$, while no other player has a best response. In the network that results if 1 replaces link $\overleftarrow{12}$ by $\overleftarrow{13}$, player 1's best response is to replace $\overleftarrow{13}$ by $\overleftarrow{12}$, and no other player has a best response (for instance, player 5 *without constraints* could replace $\overleftarrow{53}$ by $\overleftarrow{54}$ or $\overleftarrow{51}$, but

[11] Under certain conditions the existence of strict Nash $\mathcal{K}$-networks is guaranteed. The following was suggested by a referee. A strict Nash network exists if a strict Nash network exists for each block of the underlying graph. (A block of a graph is a maximal biconnected subgraph of a graph, and a graph is biconnected when it does not contain any articulating node). As the referee suggested, if each block is a Hamiltonian graph (i.e., where a wheel connecting all its nodes is feasible), then a strict Nash network exists, moreover, it is efficient.

[12] An "absorbing set" of a Markov chain is a minimal set of states (in this case states = networks) such that once entered, it is never abandoned.

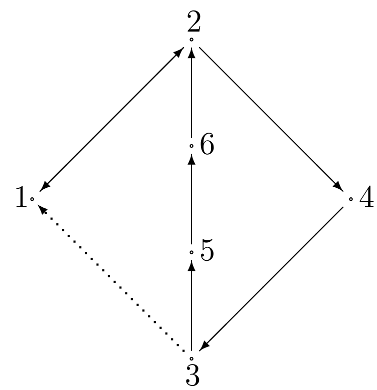

Figure 3: Example 2

these links are not admissible given 5's affiliation). In fact, the best response dynamic would oscillate between these two networks, with player 1 alternatively linking 2 and 3. Note that every player receives the same payoff in either of these networks. Note also that a strict Nash $\mathcal{K}$-network exists: $g_{12} = g_{24} = g_{46} = g_{65} = g_{53} = g_{31} = 1$, and $g_{ij} = 0$ otherwise.

One might expect a similar result here to the one obtained for the two-way flow model: convergence of the dynamic process to strict or quasi-strict Nash $\mathcal{K}$-networks. Once again another difference with the two-way flow model is encountered: in the one-way flow model it may be the case, as is for certain covers, that no miscoordination-proof network exists. It can be checked that this is the case for Example 1: for this societal cover, starting from any $\mathcal{K}$-network, best response would keep oscillating like a kaleidoscope, connecting and disconnecting the network and never reaching a strict Nash or a quasi-strict Nash $\mathcal{K}$-network, which means in particular that *convergence is not even guaranteed in terms of payoffs.*[13] With this general result about convergence discarded, each societal cover is a case-study. Although the issue of convergence when strict Nash $\mathcal{K}$-networks exist remains open, we have established it for the tree types of cover for which we show the existence of strict Nash $\mathcal{K}$-networks.

Proposition 4 *Let $\mathcal{K}$ be a societal cover, if $\mathcal{K}$ is (i) linear or (ii) in wheel or (iii) its core contains at least as many players as there are societies in $\mathcal{K}$, then Bala and Goyal's best response dynamic model converges to a strict Nash $\mathcal{K}$-network, with probability* 1. *In cases (i) and (iii) convergence to an efficient $\mathcal{K}$-network is guaranteed, while in case (ii) it is not guaranteed.*

[13] It is immediate to prove that starting from any $\mathcal{K}$-network a Nash, i.e., minimally connected, $\mathcal{K}$-network is reached with probability 1. An exhaustive listing of all Nash $\mathcal{K}$-networks for this cover is possible and shows that from any of them a sequence of miscoordinated best responses may disconnect the network.

The proof consists of an adaptation of Bala and Goyal's proof of their Theorem 3.1, which requires all the complications that a societal cover entails to be overcome. It is given in the Appendix and illustrates how these complications can, at least in these cases, be circumvented.

5 Concluding remarks

Olaizola and Valenciano (2011) studies the impact of institutional constraints, as modeled by a societal cover, on Bala and Goyal's (2000) benchmark two-way flow model. This paper addresses a similar study for their other benchmark model: the one-way flow model. The table below summarizes the main results, stressing both the parallelisms and the significant differences in the models in the context of link-formation constrained by a societal cover. If center-sponsored stars as strict Nash networks generalize to oriented trees (perhaps "grafted" when there are "hinge-players", i.e., individuals who are the only one at the intersection of the reaches of some other two) in the context of $\mathcal{K}$-networks, wheels must generalize to possibly interconnected wheels in this context when there are such hinge-players. But, in contrast with the two-way flow model, in the one-way flow model strict Nash $\mathcal{K}$-networks *may not exist*. Again in contrast with the two-way flow model, where efficiency and Nash stability are equivalent, in the one-way flow model even strict Nash $\mathcal{K}$-networks may be inefficient. Finally, Bala and Goyal's dynamic model, which in the two-way flow model never fails to converge to a strict or quasi-strict Nash $\mathcal{K}$-network, may in the one-way flow model fail to converge to a quasi-strict Nash $\mathcal{K}$-network, thus failing to converge even in terms of payoffs.

$\mathcal{K}$-network	**Two-way flow model**	**One-way flow model**
Nash	= minimally connected	= minimally connected
Efficient	Efficient = Nash	Nash *not* always efficient
Strict Nash (S.N.)	always exist	may *not* exist
S. N. architectures	oriented (grafted) trees	(interconnected) wheel(s)
Hinge players	necessary for grafted	necessary for multiple wheels
Dynamics' convergence	always to S.N./quasi-S.N.	may *not* converge

In short, the impact of introducing the constraint of a societal cover seems to be considerably greater on the results for the one-way flow model than for the two-way flow model. Needless to say, if such difficulties arise when no friction is assumed in the flow of information through the network, one can only expect further difficulties with the introduction of decay. In fact, in the presence of decay we have not been able to obtain any useful result.

Appendix

Proof of Proposition 4: (i) Consider first the case of a linear cover with two societies, $\mathcal{K} = \{A, B\}$, whose intersection is non-empty. We prove that starting from any $\mathcal{K}$-network g there is a positive probability of transiting to a strict Nash $\mathcal{K}$-network (absorbing state in the Markov process) in finite time. As there is a positive probability at each period that all but one agent will exhibit inertia, it suffices to see that starting from any $\mathcal{K}$-network there is a finite sequence of players' best responses that leads to a strict Nash $\mathcal{K}$-network. To that end, take a player i_1 in $A\backslash B$, let him/her play a best response and let g^1 denote the resulting $\mathcal{K}$-network where i_1 observes at least all players in A.

First step: Form a wheel containing all nodes in A.
Take a player i_2 furthest away in A from i_1 in g^1 (i.e., the/a player in A for which the length of the shortest path $i_2 \overset{g^1}{\rightarrow} i_1$ is the greatest). This means that i_1 observes all players in A without using any of i_2's links, otherwise a player other than i_2 would be furthest away in A from i_1. There is then a best response of i_2 where $\overleftarrow{i_2 i_1}$ is the only link with a player in A (if $i_2 \in A \cap B$ such a best response may include some other links with player in $B\backslash A$). Let i_2 play that best response (or "play inertia" if it was his/her current strategy [14]), and let g^2 denote the resulting $\mathcal{K}$-network. Note that in g^2 node i_1 still observes all nodes in A. Now we describe the induction step from a current network g^k and a sequence of players in A, $i_1, i_2, .., i_k$, such that each player i_r $(r = 2, .., k)$ supports with i_{r-1} the only link with players in A, and i_1 observes all players in A. Let i_{k+1} be the player furthest away in A from i_k in g^k. Again, this means that i_{k+1} observes all players in A without using any of i_k's links. There is then a best response of i_{k+1} in which his/her only link with a node in A is the link with i_k, and i_1 still observes all nodes in A. Repeat until $\{i_0, i_1, i_2, .., i_k\} = A$. At this stage agent i_1 observes all nodes in $A \cup B$ and he/she is the only node in A with possibly more than one link, then a best response of his/her is to form only one link with i_k and delete all others. At the end of this all players in A form a wheel and observe all players in $A \cup B$.

Second step: Form an "8" consisting of this wheel containing all players in A and another one containing all players in $B\backslash A$ and one player in $A \cap B$.
Take a player j_1 in $A \cap B$ that is linked by a player in $A\backslash B$ (at least one must exist). Let j_2 be the player in $B\backslash A$ furthest away from j_1. Let j_2 play the best response consisting of forming a single link with j_1. By reiterating this process a sequence $j_1, j_2, .., j_l$ is formed where $\{j_2, .., j_l\} \subset B\backslash A$ and each node j_r $(r = 2, .., l)$ supports the only link with j_{r-1}, and j_1 still observes all players in $A \cup B$. Repeat until $\{j_2, .., j_l\} = B\backslash A$. At this stage player j_1 is the only one with possibly more than one link with players in $B\backslash A$ and a best response of his/her is to form a single link in $B\backslash A$ with j_l and keep the one with a player in $A\backslash B$. At the end of this we have a wheel containing all players

[14] In what follows we omit this clause as obvious when a player plays a best response.

in $B\backslash A$ and player j_1. Therefore this wheel and the one formed in the previous step form the desired "8". If $A \cap B = \{j_1\}$ the network obtained is a strict Nash $\mathcal{K}$-network, otherwise:

Third step: We show that a sequence of best responses leads to two wheels, one containing all players in A, the other containing all players in B and *both sharing a sequence of links containing all those in* $A \cap B$.

Remember that j_1 was followed (i.e., linked) by a player in $A\backslash B$. Let i_r be the first player after j_1 in the sequence in $A \cap B$, and let $i_{r'}$ be the first player after i_r in this sequence that is followed (i.e., linked) by a player in $A\backslash B$. Now we describe a sequence of best responses: first, j_2 deletes his/her link with j_1 and links with $i_{r'}$; second, the player in $A\backslash B$ that links $i_{r'}$ deletes this link and links with the player in $A\backslash B$ that is linked by i_r; third, let i_r delete his/her link and link j_1; fourth, let the player linking with j_1 delete his/her link and link with $i_{r'}$. After these four best response movements, the reader may check that we have two wheels, one containing all players in $B\backslash A$, the other containing all those in A, and both sharing the sequence $j_1, i_r, .., i_{r'}$, where each of these nodes links with the preceding one. If this sequence contains all players in $A \cap B$, we have the desired pair of wheels, otherwise, repeat these four steps starting at $i_{r'}$ instead of at j_1. By reiterating this process we obtain a pair of wheels as desired.

Fourth step: Form an all-encompassing wheel.

Relabel by $k_1, .., k_m$, the sequence consisting of all players in $A \cap B$, where each of these players links with the preceding one. Again we give a sequence of best responses. Let j be the player in $B\backslash A$ who links k_m. Let j replace his/her link with k_m by a link with k_1. We again have an "8" in which two wheels interconnect at k_1, but now all players in $A \cap B$ are consecutively linked. This allows the following sequence of best responses. Let k_2 replace his/her link with k_1 by a link with the player, say j', in $B\backslash A$ linked by k_1. Then k_1 can delete his/her link with j'. Now k_3 replaces his/her link with k_2 by a link with j', and subsequently k_2 can delete his/her link with j'. Reiterate this till k_m replaces the link with k_{m-1} by a link with j', and k_{m-1} deletes his/her link with j'. At this stage an all-encompassing wheel is formed.

Now consider a *linear* societal cover of the form $\mathcal{K} = \{A_i\}_{i=1,2,..,m}$, where for all $i = 1, 2, .., m-1$, $A_i \cap A_{i+1} \neq \varnothing$, and in all other cases two societies do not intersect. Then start at A_1, take a player in $A_1\backslash A_2$ and form a wheel containing all players in A_1 proceeding as in the first step. Then, take a player in $A_1 \cap A_2$ that is linked by a player in $A_1\backslash A_2$ and, proceeding as in the second step, form an "8" consisting of the wheel containing all players in A_1 and another containing all players in $A_2\backslash A_1$ and the player chosen in $A_1 \cap A_2$. Then, following steps 3 and 4 (unless there is a unique player in $A_1 \cap A_2$), form a wheel containing all players in $A_1 \cup A_2$. At this stage all players in this wheel observe at least all players in $A_1 \cup A_2 \cup A_3$. Iterate this process, now taking a player in $A_2 \cap A_3$, etc., until an all-encompassing wheel is completed (when no hinge-player exists) or a sequence of wheels (in this case contacting at *strong* hinge-players at the intersection of two consecutive societies) forming a strict Nash $\mathcal{K}$-network. Observe that in both cases a unique strict Nash architecture exists and therefore the dynamic process converges to efficiency.

(ii) Consider a cover *in wheel* of the form $\mathcal{K} = \{A_i\}_{i=1,2,..,m}$ s.t. for all $i = 1, 2, .., m-1$, $A_i \cap A_{i+1} \neq \varnothing$, and $A_1 \cap A_m \neq \varnothing$, and in all other cases two societies do not intersect. Now proceed as in the linear case starting at any society, say A_r. Take a player i_1 in $A_r \backslash A_{r-1}$. Let him/her play a best response. Now i_1 observes at least all players in A_r. Form a sequence of players in A_r as described inductively. Let $i_1, i_2, .., i_k$ $(k \geq 1)$ be the sequence formed up to step k. Let i_{k+1} be the player furthest away in A_r from i_k and let i_{k+1} play a best response in which his/her only link in A_r is the link with i_k, *if such a best response exists*,[15] and reiterate this step as far as such a best response exists until a wheel containing all players in A_r is formed. Otherwise (i.e., if at some point before the wheel is formed i_{k+1} has no such best response), let him/her play any best response. This best response should include a link with a player, say i'_1, either in $A_{r-1} \backslash A_r$ or $A_{r+1} \backslash A_r$. Now recommence a sequence starting at i'_1 from the current network. It can be seen that by reiterating this process a wheel will be formed including all nodes in a society,[16] unless a strict Nash $\mathcal{K}$-network is formed in the process. Without loss of generality assume that the wheel has been formed in A_1. Now proceed as in step 2, by picking a player j_1 in $A_1 \cap A_2$ linked by a node in $A_1 \backslash A_2$ and forming a sequence of players in $A_2 \backslash A_1$ as described inductively. Let $j_1, j_2, .., j_k$ $(k \geq 1)$ be the sequence formed up to step k. Let j_{k+1} be the player furthest away in $A_2 \backslash A_1$ from j_k in the current network and let j_{k+1} play a best response where the only link that he/she has in A_2 is the link with j_k, *if such a best response exists*, and reiterate this step as far as such a best response exists until a wheel containing all players in $A_2 \backslash A_1$ and node j_1 is formed. Otherwise, step 2 must be recommenced by picking a player in $A_1 \cap A_m$ linked by a player in $A_1 \backslash A_m$ until a wheel containing all players in $A_m \backslash A_1$ and the chosen one in $A_1 \cap A_m$ is formed. It can be checked that now he formation of this wheel cannot be hindered by the non-existence of the desired best response. Now proceed as in the linear case (steps 3 and 4 apply unchanged) up to the completion of a wheel or a sequence of wheels (in this case contacting at isolated players at the intersection of two consecutive societies) including all players in all societies but one, say A_r. Note that now all players except possibly some in $A_r \backslash (A_{r+1} \cup A_{r-1})$ observe all players in N. Now proceed once more as in step 2, by picking a player in $A_{r-1} \cap A_r$ linked by a player in A_{r-1} and forming a wheel with all players in $A_r \backslash (A_{r+1} \cup A_{r-1})$ and the chosen one in $A_{r-1} \cap A_r$. Then, unless $A_{r-1} \cap A_r$ is a singleton, apply steps 3 and 4 to merge this wheel with the one including all players in A_{r-1}. At this stage, some players at $A_r \cap A_{r+1}$ may support some unnecessary links with players in $A_r \backslash A_{r+1}$. Finally, let these players play best responses and delete these links, then a strict Nash $\mathcal{K}$-network consisting of either an all-encompassing wheel or a sequence of wheels containing all players is formed. Only in the first case the resulting $\mathcal{K}$-network is efficient.

(iii) Finally, if the core of a societal cover $\mathcal{K}$ contains at least as many players as there are societies in $\mathcal{K}$, it is not difficult to adapt the proof, "expanding" an initial

[15] By contrast with the linear case, this is now not guaranteed.

[16] In the worst case, after the attempt in $m-1$ societies fails.

wheel containing all players in a society. A sufficient number of players within the core ensures that all such expansions are feasible and necessarily lead to an all-encompassing wheel. Therefore the dynamic process converges to an efficient $\mathcal{K}$-network. ■

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