# Network formation under linking constraints*

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**Abstract**

We study the effects of linking constraints on stability, efficiency and network formation. An exogenous "link-constraining system" specifies the admissible links. It is assumed that each player may initiate links only with players within a specified set of players, thus restricting the feasible strategies and networks. In this setting, we examine the impact of such constraints on stable/efficient architectures and on dynamics.

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# 1 Introduction

In recent years, the study of the economics of networks has attracted considerable attention from researchers and become one of the hottest topics of economic research[1]. The economics of networks is, in Goyal's words, "*an ambitious research program which combines aspects of markets (e.g., prices and competition) along with explicit patterns of connections between individual entities to explain economic phenomena*" (Goyal, 2007, p. 6).

Several seminal papers provide the basic models of strategic formation of networks. In the simplest model, links are formed unilaterally (Goyal, 1993, Bala, 1996). In this setting, Bala and Goyal (2000a) study Nash and strict Nash stability and provide a dynamic model. A model where links are formed on the basis of bilateral agreements is studied by Jackson and Wolinsky (1996), who introduce the notion of pairwise stability. These seminal papers assume homogeneity across players and that the current network is common knowledge to all node-players. These models have been extended in different directions. Bala and Goyal (2000b) introduce imperfect reliability of links. Galeotti et al. (2006) consider heterogeneous players, while Bloch and Dutta (2009) consider endogenous link strength. The common knowledge assumption may be unrealistic in many cases, and indeed is dropped by McBride (2006), who studies the effects of limited perception, namely, assuming that each node-player perceives the current network only up to a certain distance from the node.

In the seminal models, networks provide a means for the flow of information or other benefits through the links, but the current network is assumed to be common knowledge to all players, who may unrestrictedly initiate links with any other players. This may be an unrealistic assumption in some cases and, in general, the larger the number of agents and the network are the more unrealistic it will be. Due to what can generically be referred to as "institutional constraints" (social, cultural, linguistic, geographical, economic, etc.), individuals may often see only "part of the world" and initiate links only within that part or a part of that part. Thus, it seems more realistic to assume that each individual may initiate links only with a subset of players. In a way, this is an unorthodox approach if, as put by Goyal, "*the theoretical research on network effects (..) is motivated by the idea that,* within the same group [in italics]*, individuals will have different connections and that this difference in connections will have a bearing on their behavior.*" (Goyal, 2007, p. 7). Nevertheless, this is the approach adopted here and it is worth remarking that the no-constraints assumption is in fact a particular case of the more general setting adopted here. In particular, this allows Bala and Goyal's (2000a) "two-way flow" basic model[2], on which we concentrate in this paper, to be

[1] Some recent books surveying this literature are Goyal (2007), Jackson (2008) and Vega-Redondo (2007).

[2] In this model players may initiate links unilaterally, information flows through links in both directions and a player's payoff is increasing with the number of players she is connected directly or indirectly.

integrated into a wider model which sheds new light on various conclusions of their model, showing which prevail and up to which point, and which do not in this wider setting.

Based on this idea, this paper focusses on the effects of such institutional constraints on stability, efficiency and network formation, assuming that an exogenous "link-constraining system" specifies which links are feasible, thus restricting the feasible networks. Such link-constraining system can be specified by an underlying undirected network consisting of the admissible links or equivalently by a collection of sets specifying each player's "reach", i.e., the set of players with whom she may initiate links. It is also assumed that players in the same component of the link-constraining underlying network have common knowledge of the part of the current network connecting individuals in that component. In other words, players know the part of the current network that concerns their payoff, given that only players in the same component of the underlying network may influence their payoff. Further note that this model collapses to Bala and Goyal's (2000a) unrestricted setting for the particular case in which the underling constraining network is the complete network.[3]

For any given link-constraining system, we pay only attention to the admissible networks (i.e., those consistent with it) and first extend Bala and Goyal's (2000a) notion of a Nash network as those admissible networks where no player has an incentive to change her strategy, i.e., her choice of admissible links. We then easily extend their characterization of Nash networks as those among the admissible networks which are minimally connected. The set of such Nash networks is thus a subset of the set of Bala and Goyal's unrestricted Nash networks. Then the notion of strict Nash network is also naturally extended to this setting. Now a strict Nash network is a network consistent with the link-constraining system where no player may initiate and/or delete any admissible link(s) without loss. By contrast with Nash networks, things turn out to be more complicated with *strict* Nash networks. In Bala and Goyal's setting, the center-sponsored star is the only (non-empty) architecture of strict Nash networks, while in our setting the center-sponsored star architecture is feasible only when there is at least one player that can initiate a link with any other player. Nevertheless, even when the center-sponsored star architecture is feasible, this might not be the only possible architecture of strict Nash networks. A variety of architectures of strict Nash networks appear when constraints are introduced. Nevertheless, some patterns are common to these architectures. Moreover, a full characterization of all strict Nash networks for a link-constraining system is provided by means of a condition that encapsulates synthetically the essence of the architecture of these networks, embodying

[3] A different and in a sense opposite form of link-formation constraint is studied in Haller (2012), where link formation takes place around a pre-existing core network. In our setting the feasible networks are those *contained* in the underlying one, while in Haller's model the feasible networks are those *containing* the pre-existing one. Thus, in our case the complete network yields the benchmark model, while in Haller's this is so for the empty core network. As in Haller's model the pre-existing network is not a set of forbidden links, but a set of publicly provided links, there is no further relationship with the model considered here.

a clear hierarchical principle. The main features of their architectures, where stars continue to play a prominent role, are studied. Particular attention is paid to the role of certain players who are the unique players in the intersection of other players' reach, by means of whom different groups can be connected. It turns out that the two-way flow model under constraints yields as strict Nash networks the paradigm of hierarchical structures: either oriented diverging trees (also called "arborescences" in graph theory) or a sort of "grafted" overlapping oriented trees. The latter are proved to be possible only when there are "hinge-players", i.e., players who are the *unique* node in the intersection of other players' reach.

We then apply Bala and Goyal's dynamic model, where starting from any initial network each player with some positive probability plays a best response or randomizes across them when there is more than one, otherwise the player exhibits inertia, i.e., keeps her links unchanged. In this way, a Markov chain on the state space of all networks is defined. In Bala and Goyal's setting, the absorbing states are precisely the strict Nash networks and they prove that starting from any network the dynamic process converges to a strict Nash network (i.e., the empty network or a center-sponsored star) with probability 1. When adapted to our setting the best response dynamic model *does not* necessarily lead to strict Nash networks. The reason is that in our more complex setting this dynamic process may lead to the formation of partially stable "incomplete" strict Nash incompatible networks that cannot be part of the same strict Nash network, thus blocking the converging process. Therefore linking constraints may hinder the way towards strict Nash networks. Nevertheless, best response dynamics lead to absorbing sets of minimally connected networks that we call "*quasi-strict* Nash networks" and characterize them. Thus, with probability 1, best response dynamics would lead either to a strict Nash network (whenever the set of quasi-strict Nash networks reached is a singleton) or one of these absorbing sets of quasi-strict Nash networks where the best response dynamics would oscillate forever. Nevertheless *stability is reached in terms of payoffs* as it is proved that all quasi-strict Nash networks within each of these absorbing sets yield the same payoffs to all players.

The rest of the paper is organized as follows. In section 2, the basic model is specified along with the necessary notation and terminology. Section 3 studies stability and efficiency under link formation constraints. In section 4, Bala and Goyal's dynamic model is extended to this setting. Finally, section 5 summarizes the main conclusions and points out some lines of further research.

# 2 The model

Let $N = \{1, 2, .., n\}$ denote the set of *nodes* or *players*. Players may choose with which other players to initiate or support *links*. By $g_{ij} \in \{0, 1\}$ we denote the existence ($g_{ij} = 1$) or not ($g_{ij} = 0$) of a link connecting $i$ and $j$ initiated by $i$. Vector $g_i =$

$(g_{ij})_{j \in N \backslash i} \in \{0,1\}^{N \backslash i}$ specifies[4] the set of links supported by $i$ and will be referred to as an (unrestricted) *strategy* of player $i$. $G_i := \{0,1\}^{N \backslash i}$ denotes the set of $i$'s (unrestricted) strategies and $G_N = G_1 \times G_2 \times .. \times G_n$ the set of (unrestricted) strategy profiles. An unrestricted strategy profile $g \in G_N$ univocally determines a directed *network*[5] $(N, \Gamma_g)$, where

$$\Gamma_g := \{(i,j) \in N \times N : g_{ij} = 1\},$$

that we identify with $g$ and refer to as network $g$. If $M \subseteq N$ we denote by $g\mid_M$ the *subnetwork* $(M, \Gamma_{g|_M})$ with

$$\Gamma_{g|_M} := \{(i,j) \in M \times M : g_{ij} = 1\}.$$

A network $(N, \Gamma)$ is *non-directed* if $(i,j) \in \Gamma \Rightarrow (j,i) \in \Gamma$.

**Definition 1** *A "link-constraining system" in $N$ is a collection of subsets of $N$, $\mathcal{L} = \{\mathcal{L}_i\}_{i \in N}$, such that, for all $i$, $i \in \mathcal{L}_i$, and for all $i, j \in N$: $i \in \mathcal{L}_j$ if and only if $j \in \mathcal{L}_i$.*

By dropping the second condition, asymmetric constraining systems can be considered, but here we constrain our attention to the symmetric case specified. We refer to $\mathcal{L}_i$ as the *reach* of $i$, which represents the set of *nodes* that $i$ may directly access. Each player $i$ is assumed to be able to initiate links with any player in $\mathcal{L}_i$ different from herself, as it is only a matter of convenience to include $i$ in set $\mathcal{L}_i$. The *hub* of a link-constraining system is the set of nodes whose reach is $N$, i.e.,

$$hub(\mathcal{L}) := \{i \in N : \mathcal{L}_i = N\}.$$

This set may be empty.

Note that a link-constraining system $\mathcal{L}$ can be equivalently specified by the non-directed *underlying network* $g_{\mathcal{L}} = (N, \Gamma(\mathcal{L}))$, where

$$\Gamma(\mathcal{L}) := \{(i,j) \in N : i \in N, j \in \mathcal{L}_i \backslash i\},$$

which specifies the admissible links.

Given a link-constraining system $\mathcal{L}$ a *society* is a maximal set of nodes $A$ such that for all $i, j \in A$, $i \in \mathcal{L}_j$. That is, a society is a maximal set of players where any two players are within each other's reach. Therefore it is within societies that links may be formed. Let $\mathcal{K}(\mathcal{L})$ be the collection of all these (possibly overlapping) maximal sets that derive from $\mathcal{L}$, whose union is obviously $N$. In fact, $\mathcal{L}$, $g_{\mathcal{L}}$ and $\mathcal{K}(\mathcal{L})$ encapsulate the same information in three different ways. Some results can be stated in a simpler and clearer way in terms of societies. In particular all examples of constraining systems in this paper are expressed in these terms, much easier to represent graphically.

The following definition constrains the structure of a network so as to be consistent with a given link-constraining system in $N$ by ruling out links that are not feasible.

[4] We always drop the brackets "{..}" in expressions such as $N \backslash \{i\}$.

[5] In graph theory this is called a "digraph" without loops, i.e., edges connecting a node with itself (see, for instance, Tutte (1984)).

**Definition 2** *A network $g$ is consistent with a link-constraining system $\mathcal{L}$ (or is an $\mathcal{L}$-network) if for every link $g_{ij} = 1$, $j \in \mathcal{L}_i$.*

A vector $g_i = (g_{ij})_{j \in \mathcal{L}_i \setminus i} \in \{0,1\}^{\mathcal{L}_i \setminus i}$ specifies a set of $\mathcal{L}$-*feasible* links initiated by $i$ and is referred to as an $\mathcal{L}$-*admissible strategy* of player $i$, as we assume $i$'s capacity to choose which links to support *in* $\mathcal{L}_i$. $G_i(\mathcal{L}) := \{0,1\}^{\mathcal{L}_i \setminus i}$ denotes the set of $i$'s $\mathcal{L}$-admissible strategies and $G_{\mathcal{L}} = G_1(\mathcal{L}) \times G_2(\mathcal{L}) \times .. \times G_n(\mathcal{L})$ the set of $\mathcal{L}$-admissible strategy profiles. An $\mathcal{L}$-admissible strategy profile $g$ univocally determines an $\mathcal{L}$-network that we identify with $g$.

Given a network $g$, we denote $\bar{g}_{ij} := max\{g_{ij}, g_{ji}\}$. In this way a non-directed network $\bar{g}$ is defined[6]. It is assumed that each node contains valuable information and a link allows that information to flow in both directions, without friction or decay, independently of who supports it, so that each node receives the information from all nodes with which it is connected by a path. Then, $\bar{g}$ represents the effective communication provided by network $g$, which is independent of who supports the existing links according to the assumptions of the model. We say that there is a *path* of *length* $k$ from $i$ to $j$ in $g$ if there exist $k+1$ players $j_0, j_1, .., j_k$, s.t. $i = j_0$, $j = j_k$, and for all $l = 1, .., k$, $\bar{g}_{j_{l-1}j_l} = 1$, and we say that such a path is $i$-*oriented* if for all $l = 1, .., k$, $g_{j_{l-1}j_l} = 1$. A path (oriented or not) is $\mathcal{L}$-feasible if all its links are $\mathcal{L}$-feasible. The set of players with whom $i$ supports a link is denoted by $N^d(i; g)$, and the set of players connected with $i$ by a path (union $\{i\}$) by $N(i; g)$, and their cardinalities by $\mu_i^d(g) := \# N^d(i; g)$ and $\mu_i(g) := \# N(i; g)$. We say that a network $g$ is an *oriented diverging tree* (*converging tree*) if there is a node $i_0$ such that for any other node $j$ there is a unique path connecting it with the *node root* $i_0$ and such path is $i_0$-*oriented* ($j$-*oriented*).

A *component* of a network $g$ is a subnetwork $g\mid_C$, where $C \subseteq N$, such that any two players in $C$ are connected by a path, and no player in $N \setminus C$ is connected by a path with a player in $C$. We say that $g$ is *connected* if $g$ is the unique component of $g$. A network is *minimal* if for all $i, j$ s.t. $g_{ij} = 1$, the number of components of $g$ is smaller than the number of components of $g - ij$, where $g - ij$ is the network that results by replacing $g_{ij} = 1$ by $g_{ij} = 0$ in $g$. A network is *minimally connected* if it is connected and minimal.

Given a link-constraining system $\mathcal{L}$ and the set of nodes $C$ in a component of the underlying network $g_{\mathcal{L}}$, $\mathcal{L}(C) = \{\mathcal{L}_i\}_{i \in C}$ is called a *component* of $\mathcal{L}$. A link-constraining system $\mathcal{L}$ is *connected* if its underlying network $g_{\mathcal{L}}$ is connected, i.e., $\mathcal{L}$ is the unique component of $\mathcal{L}$.

We denote by $g_{-i}$ the network where all links supported by $i$ in $g$ are deleted, and by $(g_{-i}, g_i')$ the strategy profile and network that results by replacing $g_i$ by $g_i'$ in $g$.

Let $v > 0$ be the payoff that player derives from connecting directly (by a link) or indirectly (by a path) with another player, and $c > 0$ the cost for a player of initiating a link. We assume $v > c$, so that connections with *new* nodes are always profitable[7].

[6] In graph theory terms, $\bar{g}$ is the "underlying graph" of digraph $g$ (see, e.g., Tutte, 1984).

[7] Most of the results presented here can easily be extended with slight modifications to the case where payoffs are, as in Bala and Goyal (2000a), given by a function $\Phi(\mu_i(g), \mu_i^d(g))$, where $\Phi(x, y)$

Thus, the payoff of player $i$ in $g$ is

$$\Pi_i(g) = \sum_{j \in N(i;g)} v - \sum_{j \in N^d(i;g)} c,$$

that is to say:

$$\Pi_i(g) = v\mu_i(g) - c\mu_i^d(g). \tag{1}$$

An $\mathcal{L}$-network is *efficient* if it maximizes the aggregate payoff under the constraint of $\mathcal{L}$-feasible payoffs, that is, those that can be obtained by means of $\mathcal{L}$-networks.

We next discuss some notions of stability of networks consistent with a given link-constraining system $\mathcal{L}$.

# 3 Stability and efficiency

The following definitions are natural extensions of the notions of Nash stability and strict Nash stability following Bala and Goyal (2000a) for an $N$-network in a scenario where payoffs are given by (1) and: (i) a link-constraining system $\mathcal{L}$ specifies the admissible links in $N$, and (ii) all players in the same component of $\mathcal{L}$ have common knowledge of the part of the current network connecting individuals in that component. The common knowledge assumption restricted to players in the same component of $\mathcal{L}$ can be justified by assuming that information about the current network propagates through the underlying network $g_{\mathcal{L}}$[8]. Note that this scenario yields the unconstrained and common-knowledge environment of Bala and Goyal (2000a) for the particular case of the trivial link-constraining system $\mathcal{L}_i = N$ for all $i$.

**Definition 3** *A Nash $\mathcal{L}$-network is an $\mathcal{L}$-network $g$ that is stable under $\mathcal{L}$-admissible strategies, that is, for all $i \in N$:*

$$\Pi_i(g) \geq \Pi_i(g_{-i}, g'_i) \quad \textit{for all } g'_i \in G_i(\mathcal{L}). \tag{2}$$

When (2) holds, we say that $g_i$ is a *best (admissible) response* of $i$ to $g_{-i}$. Thus, in a Nash $\mathcal{L}$-network every player is playing a best $\mathcal{L}$-admissible response to those played by the others.

The stability notion can be refined in the strict sense by extending Bala and Goyal's strict Nash networks.

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is strictly increasing in $x$ and strictly decreasing in $y$, where $x$ and $y$ are non negative integers. More precisely, all results relative to stability (in Nash or strict Nash sense) extend assuming $\Phi(x, 0) > 0$ and $\Phi(x+1, y+1) > \Phi(x, y)$. Nevertheless, we prefer this simpler assumption about payoffs so as to make the statements of the basic results simpler.

[8] This assumption can be weakened by assuming that each player knows which nodes are within her reach and the payoff associated with each of her strategies if played against the current network. This is a weaker assumption as many different networks may yield the same payoff, and it is not completely unrealistic: one individual may have a clear idea of how worthy is a connection even ignoring the details of the connections of that connection.

**Definition 4** *A strict Nash $\mathcal{L}$-network is a Nash $\mathcal{L}$-network $g$ such that for all $i \in N$:*

$$\Pi_i(g) > \Pi_i(g_{-i}, g'_i) \quad \textit{for all } g'_i \in G_i(\mathcal{L}) \ (g'_i \neq g_i). \tag{3}$$

Thus, (3) means that in a strict Nash $\mathcal{L}$-network every player is playing her *unique* best (admissible) response to those played by the others.

Given the constraints on information, strategies and feasible networks that a link-constraining system imposes, the set of players in each component $\mathcal{L}(C)$, form an entirely "separate world": no link with players outside $C$ is possible and no information about them reaches $C$. In particular we have the following straightforward result.

**Proposition 1** *An $\mathcal{L}$-network $g$ is a Nash (strict Nash) $\mathcal{L}$-network if and only if $g\,|_C$ is a Nash (strict Nash) $\mathcal{L}(C)$-network for each component $\mathcal{L}(C)$ of $\mathcal{L}$.*

**Remark:** Although isolated individuals, i.e., such that $\mathcal{L}_i = \{i\}$ are included in the model, such trivial cases are of no interest in this setting.

Therefore, in view of Proposition 1 and the preceding remark, in what follows *our attention is constrained to connected link-constraining systems without isolated individuals*. The following proposition extends Bala and Goyal's result (Proposition 4.1) to this setting.

**Proposition 2** *An $\mathcal{L}$-network $g$ is a Nash $\mathcal{L}$-network if and only if it is minimally connected.*

**Proof.** *Necessity* ($\Rightarrow$): Let $\mathcal{L}$ be a connected link-constraining system in $N$, and $g$ an $\mathcal{L}$-network. Assume $g$ is not connected. Then there exist two nodes $i, j \in N$ not connected by a path in $g$. As $\mathcal{L}$ is connected, a finite sequence of nodes $x_1, .., x_m$ exists, such that $x_1 = i$, $x_m = j$ and for each $k = 1, .., m-1$, $x_{k+1} \in \mathcal{L}_{x_k}$ (and therefore $x_k \in \mathcal{L}_{x_{k+1}}$). Then for at least two consecutive nodes among these $m$ nodes, say $x_k$ and $x_{k+1}$, there is no path in $g$ connecting them. But then it is feasible and profitable for either of these two nodes to initiate a link with the other. Thus $g$ must be connected. If $g$ were not minimal there would be some superfluous link that could be eliminated and that would benefit the player that did so, and consequently $g$ would not be a Nash $\mathcal{L}$-network.

*Sufficiency* ($\Leftarrow$): Reciprocally, assume that $g$ is minimally connected. Let $i$ be any player and $g'_i$ be any strategy $g'_i \in G_i(\mathcal{L})$ $(g'_i \neq g_i)$. We show that $\Pi_i(g) \geq \Pi_i(g_{-i}, g'_i)$. A new strategy $g'_i \neq g_i$ means deleting some links and initiating new ones. If $g$ is minimally connected, then each deletion means disconnecting $i$ with a set of nodes, and if there is more than one deletion, any two of these sets of nodes disconnected from $i$ must also be disconnected from each other (otherwise a deleted link would be redundant). Thus the number of links initiated should be at least equal to the number deleted, otherwise the payoff would decrease. But then $i$'s payoff for $(g_{-i}, g'_i)$ cannot be greater than for $g$. Therefore if $g$ is minimally connected no player has an incentive to make any $\mathcal{L}$-admissible change. ∎

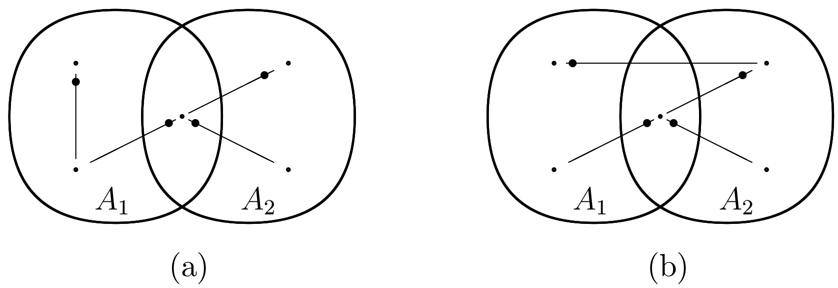



Figure 1: Minimally connected networks and $\mathcal{L}$-networks.

In Bala and Goyal (2000a, Proposition 4.3), the following result is established (in our terminology and under the assumptions about costs and benefits made here[9]): a network is efficient if and only if it is minimally connected, and Nash networks are those minimally connected. In view of this, we have the following

**Corollary 1** *A network $g$ is an efficient $\mathcal{L}$-network if and only if $g$ is a Nash $\mathcal{L}$-network.*

Therefore, for any given set of nodes $N$ and any link-constraining system $\mathcal{L}$, the set of Nash $\mathcal{L}$-networks is a subset of the set of standard unrestricted Nash networks. In Figure 1 two minimally connected networks are represented[10] assuming that the admissible links are only those connecting individuals within the same set (society) of the two represented ($A_1$ and $A_2$): (a) is a Nash $\mathcal{L}$-network, while (b) is not even an $\mathcal{L}$-network because it contains a link that is not admissible.

We now focus on strict Nash $\mathcal{L}$-networks. Stars of different types play an important role in network stability in different contexts (see, e.g., Bala and Goyal (2000a, 2006), Jackson and Wolinsky (1996), Bloch and Dutta (2009)), and, as we show below, they are also important in connection with strict Nash $\mathcal{L}$-networks. In this context, the following variant of the notion of center-sponsored star proves useful.

**Definition 5** *A set of players $M \subseteq N$ ($\#M \geq 2$) is said to be connected by a center-sponsored star $s$ in a network $g$ if $g\mid_M = s$ and there is a node $i \in M$ s.t. $N^d(i;g) = M \backslash i$ and $g_{jk} = 0$ for all $j \in M \backslash i$ and all $k \in M \backslash j$.*

Note that, according to this definition: (i) a center-sponsored star does *not* necessarily connect all players in $N$; (ii) its center $i$ can be linked from other nodes different from those in the star; and (iii) the nodes in the periphery, i.e., those $j$ in $M$ s.t. $g_{ij} = 1$ can be connected with other nodes that do not belong to the star. When $M = N$ we say that the star is *all-encompassing*.

[9] In fact, given their weaker assumptions on the payoffs (see footnote 7), the empty network may also be Nash stable in their setting, as would be the case in ours assuming $c > v$ in (1).

[10] As in all figures, nodes are represented by dots (without labels unless convenient for the purpose of the illustration), links by segments between them, and a filled circle over a link close to a node indicates the node that supports it.

Re-stated in terms of the current setting, notation and terminology, and adapted to it, Bala and Goyal (2000a, Proposition 4.2) establish the following result: *the only strict Nash networks are those consisting of a single center-sponsored star that connects all players*[11].

As we show below, a link-constraining system diversifies the stable/efficient networks. A variety of constellations of interconnected stars emerges as possible strict Nash $\mathcal{L}$-networks depending on the structure of the link-constraining system; moreover, in general, several architectures appear as strict Nash for a given link-constraining system. Our next goal is to identify and characterize these networks.

In the characterization of strict Nash $\mathcal{L}$-networks, the following binary relation on $N$ associated with a network $g$ plays an important role. Let $\overset{g}{\to}$ be the transitive closure of the binary relation $L_g$ defined by

$$i \ L_g \ j \quad \Leftrightarrow (i = j \ \ or \ \ g_{ij} = 1).$$

That is to say, $i \overset{g}{\to} j$ if $i = j$ or there exists an *i-oriented* path from $i$ to $j$. This relation is obviously transitive, but in general, for an arbitrary network $g$, it is not complete, antisymmetric or acyclic[12]. But if $g$ is minimally connected, then $\overset{g}{\to}$ is certainly antisymmetric and acyclic (otherwise at least one link would be redundant). Thus, in view of Proposition 2, we have the following

**Lemma 1** *For any Nash $\mathcal{L}$-network $g$, the binary relation $\overset{g}{\to}$ is a partial order on $N$.*

For any Nash $\mathcal{L}$-network $g$, we use the following terminology. We say that $i$ is a *predecessor* of $j$ (and that $j$ is a *successor* of $i$) in $g$ if $i \neq j$ and $i \overset{g}{\to} j$. We say that a node is *terminal* in $g$ if it has no successors, and we say that a node is *maximal* in $g$ if it has no predecessors.

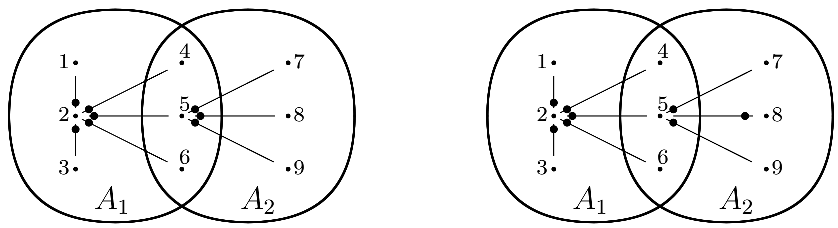
As we will presently show, *strict* Nash $\mathcal{L}$-networks have a strongly hierarchical structure, and the following terminology proves useful.

**Definition 6** *A node $j$ is "within hierarchical reach" of another node $i$ in a minimally connected $\mathcal{L}$-network $g$ if $j$ is within $i$'s reach and $j$ is not a predecessor of $i$ nor there is a predecessor of $i$ connected with $j$ through a path not containing $i$.*

That is, $j$ is within hierarchical reach of $i$ in a minimally connected network $g$ if: (i) $j \in \mathcal{L}_i \backslash i$, (ii) $j \overset{g}{\nrightarrow} i$, and (iii) there is no $k \neq i$ s.t. $k \overset{g}{\to} i$ and $j \in N(k; g \mid_{N \backslash i})$. Thus, the set of nodes within hierarchical reach of a node $i$ is obtained by discarding in the set of nodes within $i$'s reach: $i$'s predecessors and those connected with $i$'s predecessors

[11] Given their weaker assumptions on the payoffs (see footnotes 7 and 9), the empty network may also be strict Nash in their setting.

[12] A binary relation $R$ on a set $X$ is *antisymmetric* if, for all $x, y \in X$, $xRy$ and $yRx$, implies $x = y$; and $R$ is said to be *acyclic* if there is no finite chain $x_1, x_2, .., x_n$ in $X$ s.t. $x_k R x_{k+1}$ for $k = 1, 2, .., n-1$, and $x_n R x_1$, unless $x_k = x_{k+1}$ for $k = 1, 2, .., n-1$. In general, the second condition is weaker than the first, but when the relation is transitive they are equivalent.




(a) (b)

Figure 2: Hierarchical and non-hierarchical networks.

by a path not containing $i$. In particular, if a player $i$ has no predecessors, the nodes within $i$'s hierarchical reach are all those within $i$'s reach.

Note that a necessary condition for $j$ to be within hierarchical reach of $i$ in $g$ it is that $g_{ji} = 0$, but it is *not* required that $g_{ij} = 1$. When this is required, so that every node supports links with every node within its hierarchical reach, the network adopts a strongly hierarchical structure as we will presently see. This motivates the following

**Definition 7** *An $\mathcal{L}$-network $g$ is "hierarchical" if it is minimally connected and every node supports links with all those within its hierarchical reach in $g$.*

The examples in Figure 2 illustrate Definitions 6 and 7. It represents two minimally connected 9-node networks for a two-society constraint. In (a) every player supports links with all nodes within her hierarchical reach. For instance, 4 does not support any link, but there is *no* node within 4's hierarchical reach because, even though all nodes are within 4's reach, all of them are either 4's predecessors (in this case only node 2) or connected with 4's predecessors by paths not containing 4. As to the others, for instance, 5 supports links with the only three within 5's hierarchical reach (7, 8 and 9), given that the others within 5's reach are either 5's predecessors (in this case node 2) or connected with 5's predecessors by paths not containing 5. In fact, the reader may check that this condition holds for *every* node and consequently it is a hierarchical network, while it is enough inverting the direction of any single link to make the network non hierarchical. For instance, consider (b), where player 8 is paying her link, then 8 has no predecessors, so all nodes within 8's reach (i.e. all others in $A_2$) are within 8's hierarchical reach, but 8 only supports one link (with 5). Thus (b) is *not* hierarchical.

Then we have the following characterization: strict Nash $\mathcal{L}$-networks are just hierarchical $\mathcal{L}$-networks.

**Theorem 1** *A network $g$ is a strict Nash $\mathcal{L}$-network if and only if $g$ is a hierarchical $\mathcal{L}$-network.*

**Proof.** Necessity ($\Rightarrow$): Obviously, a strict Nash $\mathcal{L}$-network $g$ is also a Nash $\mathcal{L}$-network and, by Proposition 2, necessarily minimally connected, so that, by Lemma 1, $\overset{g}{\to}$ is a partial order. Now let $i$ be a node in $g$ and assume $g_{ij} = 0$ for some $j$ within $i$'s hierarchical reach, i.e., some $j \in \mathcal{L}_i \backslash i$ that is not a predecessor of $i$ and for which there is no $k$ predecessor of $i$ such that $j \in N(k; g\mid_{N\backslash i})$. As $g$ is minimally connected, there must be a path connecting $i$ and $j$, that then does not contain any predecessor of $i$. Therefore the first link on that path must be a link supported by $i$. But then $i$ can delete that link and initiate a link with $j$ without altering $i$'s payoff, and consequently $g$ is not a strict Nash $\mathcal{L}$-network.

Sufficiency ($\Leftarrow$): Assume that $g$ is a minimally connected $\mathcal{L}$-network. According to Proposition 2, $g$ is a Nash $\mathcal{L}$-network. Let $i$ be any node and any $g'_i \in G_i(\mathcal{L})$ s.t. $g'_i \neq g_i$. We show that $\Pi_i(g) > \Pi_i(g_{-i}, g'_i)$ if $g$ is hierarchical. Reasoning as in Proposition 2, as $g$ is minimally connected, $g'_i \neq g_i$ involves deleting some links and initiating at least an equal number of new links for $(g_{-i}, g'_i)$ to be also minimally connected, otherwise $i$'s payoffs would be smaller in $(g_{-i}, g'_i)$, but in fact the number of links deleted and that of those newly initiated by $i$ should be the same for the same reason. Let link $ii'$ be one of the former (i.e., $g_{ii'} = 1$ and $g'_{ii'} = 0$) and let $ij$ be one of the latter (i.e., $g_{ij} = 0$ and $g'_{ij} = 1$). If $g$ is hierarchical, either $j$ is a predecessor of $i$ in $g$ or there exists a $k$ predecessor of $i$ in $g$ such that $j \in N(k; g\mid_{N\backslash i})$. But this implies a cycle in $(g_{-i}, g'_i)$. The reason is this: evidently adding link $g'_{ij} = 1$ to $g$ means a cycle in $(g - ii') + ij$, but it must be proved that this cycle is contained *in* $(g_{-i}, g'_i)$. This is so because no link in the path in $g$ connecting $i$ and $j$ can have been initiated by $i$ (this would imply a cycle in $g$, which is assumed to be minimally connected). Therefore, no matter which other links in $g_i$ are deleted in $g'_i$, the cycle is entirely contained in $(g_{-i}, g'_i)$. The same can be said about all new links in $g'_i$ w.r.t. $g_i$, all new links are redundant in $(g_{-i}, g'_i)$. Therefore necessarily $\Pi_i(g) > \Pi_i(g_{-i}, g'_i)$. ■

As an immediate corollary of Theorem 1, we have the following conclusion that yields Bala and Goyal's result (Proposition 4.2) as a particular case.

**Corollary 2** *An all-encompassing star is a strict Nash $\mathcal{L}$-network if and only if the hub of $\mathcal{L}$ is non-empty and the center belongs to it. In particular, when $\mathcal{L} = \{N\}$ the only strict Nash $\mathcal{L}$-networks are the all-encompassing center-sponsored stars.*

**Proof.** First statement is evident: for an all-encompassing star to be feasible its center must belong to the hub, and any feasible all-encompassing star is obviously hierarchical. Assume now that $\mathcal{L} = \{N\}$ and $g$ is a strict Nash $\mathcal{L}$-network. By Proposition 2, $g$ must be minimally connected. If $g$ is not an all-encompassing star, there must necessarily exist three players $i, j, k$ such that either $g_{ji} = g_{ki} = 1$, or $g_{ij} = g_{jk} = 1$, but these are the two situations considered in Figure 2 that cannot occur in a hierarchical $\mathcal{L}$-network if $\mathcal{L} = \{N\}$. ■

This characterization allows also for a constructive proof of existence of strict Nash $\mathcal{L}$-networks for any link-constraining system $\mathcal{L}$: start at any node $i_0$ and initiate links with all nodes in $\mathcal{L}_{i_0}$, then extend the network by initiating new links from those nodes, always respecting hierarchical priority. In fact we have the following result:

**Proposition 3** *For any link-constraining system $\mathcal{L}$ and any node $i_0 \in N$ there exists an oriented diverging tree $g$ rooted at $i_0$ that is a strict Nash $\mathcal{L}$-network.*

**Proof.** Iterate the following procedure:

- Step 0: Initially let $i_0$ be any player in $N$, and $g^0$ the $\mathcal{L}$-network that results by $i_0$ initiating links with all players in $\mathcal{L}_{i_0}$, and let $C_0 := \mathcal{L}_{i_0}$.

- Step from $k$ to $k+1$: If $g^k$ is the current $\mathcal{L}$-network resulting from step $k$, take a terminal node, say $i_{k+1}$, in $g^k$, for which the set of nodes $\mathcal{L}_{i_{k+1}} \backslash C_k$ is *not empty*, and let $i_{k+1}$ initiate links with all those players. If no such node exists, stop; otherwise, let $g^{k+1}$ be the $\mathcal{L}$-network that results by adding all these links initiated by $i_{k+1}$ to $g^k$, and $C_{k+1} := C_k \cup \mathcal{L}_{i_{k+1}}$.

It is clear that if $\mathcal{L}$ is connected, this iterated process must stop in a finite number of steps (when $C_k = N$) and the resulting network will be an oriented diverging tree rooted in $i_0$ that is obviously hierarchical, thus forming a strict Nash $\mathcal{L}$-network connecting all players in $N$. ■

As a corollary of Theorem 1, the following propositions establish some prominent features of the architecture of strict Nash $\mathcal{L}$-networks that help to form a clearer idea about these networks, which we later illustrate with some examples. The first one shows the role of stars in strict Nash $\mathcal{L}$-networks.

**Proposition 4** *In a strict Nash $\mathcal{L}$-network $g$:*
*(i) There is at least one node that supports links with all nodes within its reach.*
*(ii) For each society $A \in \mathcal{K}(\mathcal{L})$, $g\mid_A$ consists of disjoint center-sponsored stars and/or isolated nodes.*

**Proof.** *(i)* By Lemma 1, given that $g$ is minimally connected, $\overset{g}{\rightarrow}$ is a partial order and necessarily exists at least one maximal element, i.e., with no predecessor. Let $i_0$ be a maximal element. As $i_0$ is maximal, by Theorem 1, necessarily $N^d(i_0; g) \cup i_0 = \mathcal{L}_{i_0}$, i.e., $i_0$ must support links with all nodes within its reach.

*(ii)* Let $A$ be a society in $\mathcal{K}(\mathcal{L})$. Assume that for some $i, j \in A$, $g_{ij} = 1$. It is enough to show that the only other link that may exist connecting any $k \in A\backslash\{i, j\}$ with $i$ or $j$ is a link supported by $i$. Assume that $g_{kj} = 1$. Then, $k$ can delete the link with $j$ and initiate one with $i$ and have the same payoff. Assume that $g_{jk} = 1$. Then, $i$ can delete the link with $j$ and initiate one with $k$ and have the same payoff. Finally, assume that $g_{ki} = 1$. Then $k$ can delete the link with $i$ and initiate one with $j$ and have the same payoff. Thus, the only remaining possibility of a link connecting any $k \in A\backslash\{i, j\}$ with $i$ or $j$ is a link $g_{ik} = 1$. ■

Observe the similarity of the proof of part *(ii)* with Bala and Goyal's proof of their result and its differences: minimal connectedness and "strict Nash-ness" do *not* entail *all* nodes in a society $A$ being connected by a *single* star. Now the possibility of other center-sponsored stars within a society is left open, along with even the possibility of some nodes being left outside these stars (but linked through nodes belonging to societies other than $A$). Yet the hierarchical arrangement of a strict Nash $\mathcal{L}$-network entails a maximum of *two* levels within each society: centers and spokes (as seen in Figure 2). The question now is: how do nodes in different societies interconnect in $g$? Evidently through overlapping societies. More precisely, the following proposition shows that in a strict Nash $\mathcal{L}$-network connections "propagate" in an oriented way that can be reversed only at a "hinge-player", i.e., a player that is linked by another two whose reaches' intersection contains only that player.

**Proposition 5** *Let $g$ be a strict Nash $\mathcal{L}$-network, and $i, j, k \in N$ $(j \neq k)$ s.t. $g_{ji} = g_{ki} = 1$, then necessarily $\mathcal{L}_j \cap \mathcal{L}_k = \{i\}$.*

**Proof.** Let $g$ be a strict Nash $\mathcal{L}$-network, and $i, j, k \in N$ s.t. $g_{ji} = g_{ki} = 1$. Assume that $i' \in \mathcal{L}_j \cap \mathcal{L}_k$, with $i \neq i'$. If $i$ and $i'$ were linked (i.e., $\bar{g}_{ii'} = 1$), then $j$ (or $k$) could delete the link with $i$ and initiate a link with $i'$ without loss. Thus, we should have $\bar{g}_{ii'} = 0$. As $g$ is minimally connected, either a path connecting $i'$ and $j$ and not containing $k$ exists, or there exists a path connecting $i'$ and $k$ and not containing $j$. In the first case $k$ can delete the link with $i$ and initiate a link with $i'$, and in the second $j$ can delete the link with $i$ and initiate a link with $i'$. In both cases this is without loss for the player who changes strategy, therefore contradicting that $g$ is a strict Nash $\mathcal{L}$-network. ■

The examples in Figure 3, where, as in all examples, the link-constraining system is represented by the associated societies, illustrate the characterization and its corollaries and convey the logic of strict Nash $\mathcal{L}$-networks. Of course, the characterizing condition of respecting hierarchical priority holds in all cases, as the reader may check. Examples (a) and (b) represent link-constraining systems with a non-empty hub where an all-encompassing center-sponsored star is *one* of the possible architectures of strict Nash $\mathcal{L}$-networks: (d) and (c) represent other strict Nash $\mathcal{L}$-networks for the same link-constraining systems. In examples (a), (b) and (d) a single center-sponsored star covers (partially) each society, while *two* center-sponsored stars cover society $A_3$ in (c) and society $A_5$ in (e), and in both cases no other link exists between pairs of individuals. In all cases, a maximal node exists (represented by a white circle "$\circ$"), but *there may be more than one,* as in examples (e), (f) and (g), which illustrate Proposition 5: stars connecting "hand in hand" by means of a "free rider" node are possible when this is the only node in the intersection of the reaches of all nodes supporting links with it. We reach in fact the following conclusion: when no pair of societies derived from the link-constraining system $\mathcal{L}$ share a single player, a strict Nash $\mathcal{L}$-network is an oriented diverging tree, as is proved by the following

**Corollary 3** *Let $\mathcal{L}$ be a link-constraining system such that for all $A, B \in \mathcal{K}(\mathcal{L})$, $A \cap B$ is empty or contains at least two nodes, then a strict Nash $\mathcal{L}$-network necessarily forms an oriented diverging tree.*

**Proof.** There is a unique path connecting any maximal node with each node. Assume that there are two maximal nodes $i_0$ and $i_1$. Then, there is a path connecting $i_0$ and $i_1$, but then there must be three nodes on that path $i$, $j$ and $k$ such that $g_{ij} = g_{kj} = 1$. Now if the intersection of any two societies in $\mathcal{K}(\mathcal{L})$ is either empty or contains *more* than a single player, this is impossible according to Proposition 5. Therefore, there

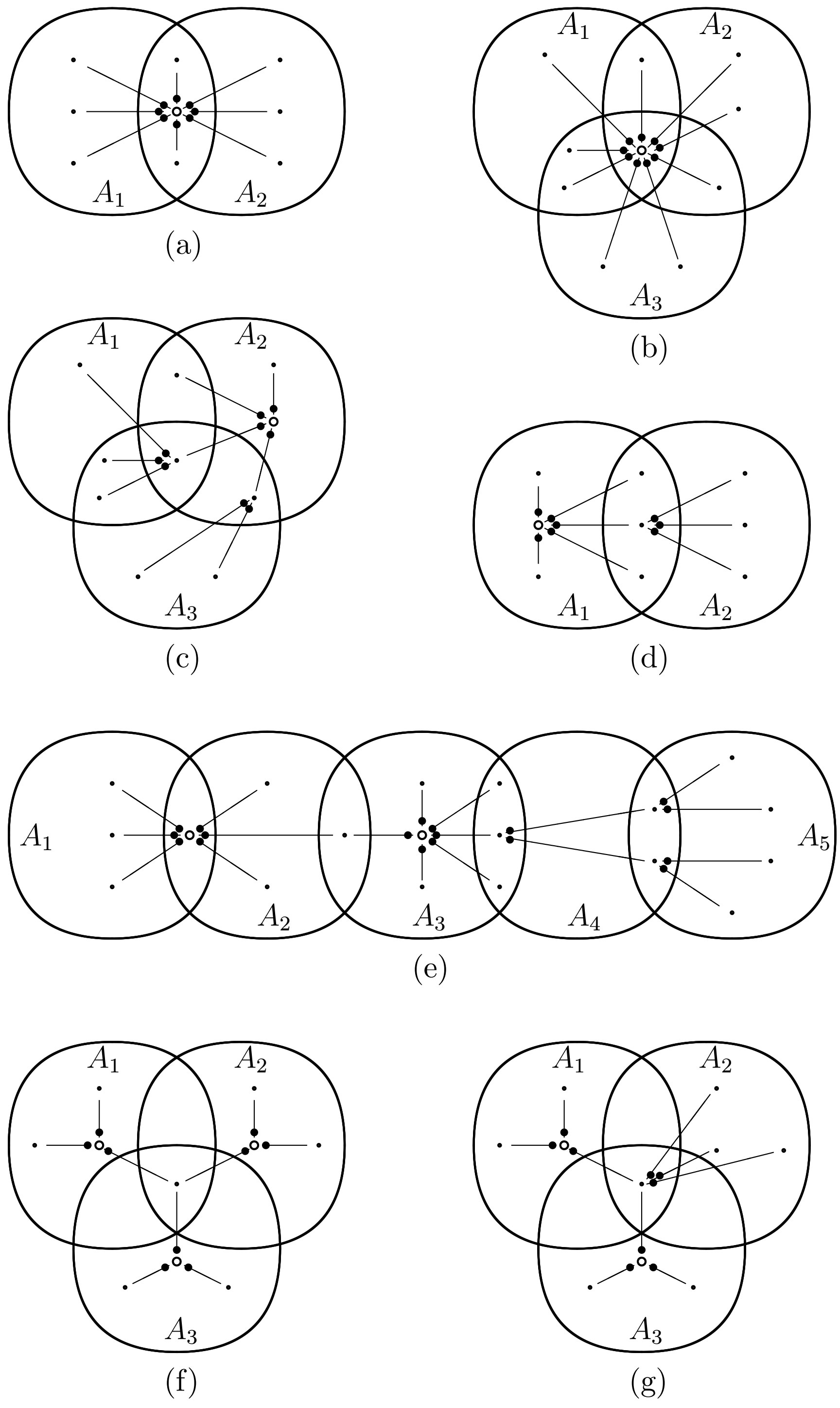



Figure 3: Strict Nash $\mathcal{L}$-networks.

can be only one maximal node connected with any other node by a unique path and consequently $g$ is an oriented diverging tree. ■

But note that, as examples (e), (f) and (g) in Figure 3 show, when there are two or more societies to which a *single* player belongs, several maximal nodes may exist. In this case, two or more "grafted" oriented diverging trees may emerge, so that any node is connected by an oriented diverging tree with at least one but possibly more maximal nodes. In this case several hierarchies may overlap consistently, e.g., in (g) two hierarchies, rooted at the two maximal nodes in $A_1$ and $A_3$, overlap over the star that connects nodes in $A_2$.

We conclude the discussion by providing some examples that illustrate the sensitivity of strict Nash architectures to small changes in the underlying constraining network and studying the strict Nash architectures for some simple classes of underlying networks of feasible links.

Consider first the *impact of eliminating a single link* $(i, j)$ from the set of feasible ones. As an immediate consequence, the set of strict Nash networks consisting of oriented diverging trees rooted at $i$ or at $j$ are different in either case. For a clear example that illustrates this, consider Bala and Goyal's setting, where $\mathcal{L}_i = N$, for all $i$ (i.e., there is a single society $K(\mathcal{L}) = \{N\}$), where center-sponsored stars are the only strict Nash networks. If a link $(i, j)$ is eliminated from the set of feasible ones, in the resulting organization into *two* societies, $N \backslash i$ and $N \backslash j$: first, all center-sponsored stars centered at $i$ or at $j$ become unfeasible; second, all diverging trees rooted at $i$ $(j)$ where $i$ supports links with all others but $j$ $(i)$ and one of them supports a link with $j$ $(i)$ appear as *new* strict Nash networks. Looking at it in the opposite direction, adding a feasible link has always an impact that can be considerable. Consider a link constraining system for which a "graft" at a node $i$ occurs in a strict Nash network. By Proposition 5, $i$ must be the unique node within reach of the two, say $j$ and $k$, supporting links with it. But as soon as link $(j, k)$ becomes feasible such network is not strict Nash any more.

As to efficiency, in view of Corollary 1 all Nash networks are efficient. Thus, the elimination of feasible links reduces the set of strict Nash networks, but *as far as this elimination does not disconnect the underlying network* the aggregate utility remains unchanged, while when the elimination disconnects the underlying network evidently the aggregate utility diminishes.

Now based on the characterizing Theorem 1 and its implications, particularly Propositions 4 and 5, we study the strict Nash networks for some simple underlying networks. Consider first, the *line*, that is, if $N = \{1, 2, .., n\}$ assume that only links between consecutive nodes are feasible. In this case every node except 1 and $n$ can be a hinge-player, thus Nash and strict Nash networks coincide and consist of all feasible networks where any two consecutive nodes are linked by a link supported by any of them and only one. Now consider the *circle*, that is, if $N = \{1, 2, .., n\}$ only links between consecutive nodes and between 1 and $n$ are feasible. In this case *all* nodes can become a hinge-player, and the architecture of Nash networks is the same as in the case of the line, with the only difference that now no node is forced to be at the extreme. But now not all Nash networks are strict, only those where the nodes at the extremes do not support the link that connects them, given that a node at an extreme that pays its link would be indifferent between its two feasible links. Finally, consider the case where the underlying $N$-network $\Gamma(\mathcal{L})$ consists of two cliques, one connecting every pair of nodes in $P$ and another those in $Q$, connected either by a 1-clique sum, i.e.,"glued together" by a node, or by a link. In the first case, the constraint in terms of societies is very simple $K(\mathcal{L}) = \{P, Q\}$, so that $P \cap Q$ is a singleton and the possible architectures of a strict Nash network are: (i) a center-sponsored star centered at this singleton; (ii) an oriented tree rooted at a node different from this singleton supporting links with all nodes within its society, and the singleton supporting links with all the rest in the other society; (iii) two center-sponsored stars, one centered at a node in $P\backslash i$ and another centered at a node in $Q\backslash i$, with $i$ playing the role of hinge player. The second case, when a single link between two nodes, say $i, j$, connects both cliques in the underlying network, in terms of societies means $K(\mathcal{L}) = \{P, Q, \{i, j\}\}$, with $i \in P$ and $j \in Q$. There are basically 4 possible architectures (8 interchanging $i$'s and $j$'s roles in the following description): (i) a player in $P\backslash i$ supports links with all other nodes in $P$, $i$ supports a link with $j$, and $j$ supports links with all other nodes in $Q$; (ii) a player in $P\backslash i$ supports links with all other nodes in $P$, $i$ supports a link with $j$, and a player in $N\backslash j$ supports links with all other nodes in $Q$; (iii) $i$ supports links with all other nodes in $P$ and with $j$, and $j$ supports links with all other nodes in $Q\backslash j$; (iv) $i$ supports links with all other nodes in $P$ and with $j$, and a player in $Q\backslash j$ supports links with all other nodes in $Q$. Observe that in cases (i) and (iii) we have a single root-oriented tree, while in (iii) and (iv) we have two grafted trees with $j$ playing the hinge player role.

Finally, in the spirit of the "community detection" problem (see, e.g., Jackson, 2008), we address a reciprocal issue to that considered so far. Given a network $g$, can it be interpreted as a strict Nash $\mathcal{L}$-network for any particular link-constraining system $\mathcal{L}$? In this respect, it is natural to express the link-constraining system in terms of its associated societies. It is easy to see that this question admits many answers: in general, an oriented diverging tree (or several grafted ones) can be seen as a strict Nash $\mathcal{L}$-network for different link-constraining systems. Restricting attention to oriented diverging trees, the following associated link-constraining systems are worth noting. Let $g$ be an oriented diverging tree rooted at $i_0$. The *generational link-constraining system*, consisting of a minimal number of societies, each consisting of all nodes at the same distance from the root that are not terminal along with their "offspring"; the *family link-constraining system* where each node forms a society with its offspring; and the trivial *binary link-constraining system* where any two directly linked nodes form a society. For all three link-constraining systems, the oriented diverging tree $g$ is a strict Nash $\mathcal{L}$-network and it is the only one with maximal node $i_0$ for the latter two.

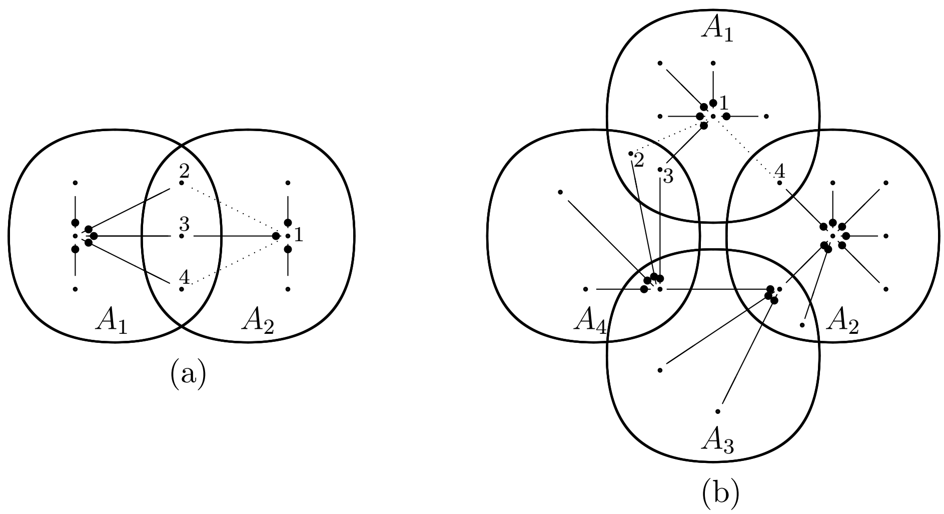



Figure 4: Dynamic deadlock towards a strict Nash $\mathcal{L}$-network.

# 4 Dynamics

We now study Bala and Goyal's (2000a) dynamic model in this setting. Namely, starting from any initial $\mathcal{L}$-network $g$, at each period, each player $i$, independently, with some positive probability responds with an $\mathcal{L}$-admissible best response[13] to $g_{-i}$ or randomizes across them when there are more than one, otherwise player $i$ exhibits *inertia*, i.e., keeps her links unchanged. In this way, a Markov chain on the state space of all $\mathcal{L}$-networks is defined. Bala and Goyal (Theorem 4.1) prove that in their setting, i.e., for $\mathcal{L} = \{N\}$, starting from any network, the dynamic process converges to a strict Nash network (i.e., the empty network or a center-sponsored star) with probability 1. In other words, the only absorbing sets are singletons consisting of strict Nash networks. The following example shows that this *is not* the case for the same dynamic model in the context of $\mathcal{L}$-networks.

**Example** In Figure 4 (a) players in $A_1$ have no best response but keep their strategies, while player 1 is indifferent between initiating a link with 2 or 3 or 4. Consequently the best response dynamic process would oscillate forever within this three-element absorbing set. Similarly, in Figure 4 (b) all players in $A_2$, $A_3$ and $A_4$ keep their strategies, while player 1 is indifferent between supporting a link with 2 or 3 or 4, and consequently best response dynamics would oscillate forever among these three networks forming a three-element absorbing set. Note that in both examples the $\mathcal{L}$-networks among which the best response dynamics oscillate are minimally connected and yield the same payoffs to all players.

The example shows an interesting difference with respect to Bala and Goyal's set-

[13] Note that if $g$ is a Nash $\mathcal{L}$-network any $\mathcal{L}$-admissible strategy $g'_i$ of player $i$ such that $\Pi_i(g) = \Pi_i(g_{-i}, g'_i)$, is a best response to $g_{-i}$.

ting. The same logic that in their setting leads to the absorbing strict Nash networks, in ours may also lead to the formation of interconnected center-sponsored stars, whose centers are fixed (i.e., immune to miscoordination), which are incompatible in any strict Nash $\mathcal{L}$-network. In this case, the converging process is blocked. Thus, in general, the dynamic process leads to an *absorbing set*, that is, a minimal set of $\mathcal{L}$-networks closed under best response dynamics. This raises the question about what these absorbing sets consist of. We call *quasi*-strict Nash $\mathcal{L}$-networks to those that belong to any of these absorbing sets and explore their structure. For this purpose a clear understanding of the possibility of "miscoordination" in a *minimally connected* $\mathcal{L}$-network is needed.

**Definition 8** *A minimally connected $\mathcal{L}$-network is said to be "miscoordination-proof", if no sequence of best responses according to the described dynamics exists which starting from that network yields a non connected $\mathcal{L}$-network.*

In a minimally connected $\mathcal{L}$-network, a best response played by a single player cannot disconnect the network, but miscoordination may occur when there exist best responses that disconnect the network when they are played simultaneously. Observe that both examples in Figure 4 consist of miscoordination-proof $\mathcal{L}$-networks.

In general, miscoordination may occur in a single best response step when two nodes support links with the same node $k$ (or two different nodes $k, k'$ connected by a path) and both have best responses that consist of breaking their links with $k$ (or $k$ and $k'$) and replacing them by initiating new ones with nodes connected by some path with the other that separately would not disconnect the network, but when they are simultaneous this would disconnect it. Moreover, even if this situation does not occur, it may be the case that this occurs after a best response step. The following lemma specifies in detail the conditions under which *none of these situations may occur in a minimally connected* $\mathcal{L}$-network, which is therefore miscoordination-proof.

**Lemma 2** *A minimally connected $\mathcal{L}$-network $g$ is miscoordination-proof if and only if (i) for every society $A \in \mathcal{K}(\mathcal{L})$, $g\mid_A$ consists of center-sponsored stars and/or isolated nodes, and (ii) for any two nodes $i, j$ and any best responses $g'_i$ and $g''_j$, such that $g^*_{ik} = g^*_{jk'} = 1$, where $g^* = ((g_{-i}, g'_i)_{-j}, g''_j)$ (i.e., $g^*$ is the network that results from $g$ when $i$ and $j$ play $g'_i$ and $g''_j$), and nodes $k, k'$ are connected by a path in $g$ not containing $g^*_{ik}$ or $g^*_{jk'}$ the following condition holds:*

$$\mathcal{L}_i \cap N(j; g^* - jk) = \varnothing \quad or \quad \mathcal{L}_j \cap N(i; g^* - ik) = \varnothing. \tag{4}$$

**Proof.** Necessity ($\Rightarrow$): Let $g$ be a minimally connected $\mathcal{L}$-network. First note that if for some society $A \in \mathcal{K}(\mathcal{L})$, $g\mid_A$ does not consist of center-sponsored stars and/or isolated nodes miscoordination between nodes of that society can surely disconnect the network[14]. Assume then that this condition holds. If for some pair of nodes $i, j$ the

[14] The proof is similar to that for Theorem 4.1 in Bala and Goyal (2000a).

condition specified fails to hold, it is easy to check that it is possible to disconnect the network by miscoordination in one or two best response steps.

Sufficiency ($\Leftarrow$): Let $g$ be a minimally connected $\mathcal{L}$-network for which all conditions in the lemma hold. Then it is easy to check that no sequence of best response steps can disconnect the network. ■

We have then the following result that proves that *quasi*-strict Nash $\mathcal{L}$-networks are just miscoordination-proof minimally connected $\mathcal{L}$-networks.

**Proposition 6** *Under a link-constraining system $\mathcal{L}$ the absorbing sets under best response dynamics consist of miscoordination-proof minimally connected $\mathcal{L}$-networks, and any miscoordination-proof minimally connected $\mathcal{L}$-network belongs to an absorbing set.*

**Proof.** First note that starting from any miscoordination-proof minimally connected $\mathcal{L}$-network best response dynamics cannot disconnect the network and can only yield another network satisfying the same conditions, i.e., another miscoordination-proof minimally connected $\mathcal{L}$-network where the number of links supported by each node remains unchanged. Therefore, any miscoordination-proof minimally connected $\mathcal{L}$-network along with all others that can be reached from it by best response dynamics form an absorbing set. It remains to be shown that there are no other absorbing sets. Starting from any $\mathcal{L}$-network, best response dynamics lead with probability 1 to a minimally connected $\mathcal{L}$-network $g$ such that for every society $A \in \mathcal{K}(\mathcal{L})$, $g \mid_A$ consists of center-sponsored stars and/or isolated nodes[15]. If some of the conditions of Lemma 2 does not hold, miscoordination is possible (in one or two steps) in a way that the network is disconnected ($i$ and $j$ deleting simultaneously their links with $k$) and a cycle appears. In a new best response step, one of the involved nodes, say $i$, links $k$ again and the other breaks the cycle. In this way, a new minimally connected network results where the $i$-centered star has a new spoke and one of the possibilities of miscoordination has disappeared. A sequence of best response steps that leads to a miscoordination-proof minimally connected $\mathcal{L}$-network is therefore proved to exist. ■

As a corollary, we have the following result that shows that when an absorbing set is reached, in spite of the possible perpetual oscillation, stability in terms of payoffs is reached given that all networks in the same absorbing set *yield the same payoffs to all players.*

**Corollary 4** *For any two quasi-strict Nash $\mathcal{L}$-networks $g, g'$ that belong to the same absorbing set and all $i \in N$, $\Pi_i(g) = \Pi_i(g')$.*

**Proof.** Let $Q$ be an absorbing set and $g \in Q$. As $g$ is a miscoordination-proof minimally connected $\mathcal{L}$-network, the number of links supported by each node is invariant under best response dynamics. Therefore, the payoffs must remain unchanged for all players within $Q$. ■

[15] The proof is similar to that for Theorem 4.1 in Bala and Goyal (2000a), merely respecting $\mathcal{L}$-feasibilty.

In summary, quasi-strict Nash $\mathcal{L}$-networks, i.e., the constituent of the absorbing sets of best response dynamics, are not very different from strict Nash $\mathcal{L}$-networks. They are minimally connected $\mathcal{L}$-networks consisting of interconnected stars, one or several disjoint ones in each society, where nodes support links with all nodes within their hierarchical reach with the only possible exception of some nodes that support links with only one node among several between which best response dynamics can oscillate. Thus, the architecture of quasi-strict Nash $\mathcal{L}$-networks is that of grafted trees, something that was only possible for strict Nash networks when a unique individual belonged to two different societies.

# 5 Concluding remarks

This paper is a first step of a research project to explore the impact of institutional constraints on network formation. Such constraints emerge due to social, cultural, linguistic, economic, geographic, etcetera reasons and cannot be ignored in many contexts. We study their impact, modeled by a link-constraining system, on Bala and Goyal's (2000a) benchmark two-way flow model. We characterize and study in some detail the structure of stable and efficient networks under linking constraints by extending Bala and Goyal's approach and results. In a nutshell, the conclusions when there is no decay can be synthesized by the equation:

$$\textit{Linking constraints} + \textit{Strict stability} = \textit{Hierarchical organization.}$$

Namely, if there is no decay, the all-encompassing center-sponsored star (when feasible) is no longer the only stable (in the strict Nash sense) architecture, but center-sponsored stars continue to be the basic building blocks of stable networks. Moreover, the architecture of such stable networks embodies a formal hierarchical principle that yields oriented diverging trees, the paradigm of hierarchical organization[16], or "grafted" overlapping oriented trees adapted to the constraints imposed by the link-constraining system. It is also proved that simple best response dynamics "work" basically well in this more complicated setting. They may fail to reach a strict Nash network if incompatible "incomplete" and "almost stable" hierarchical networks form, but a stable configuration of payoffs associated with an absorbing set of miscoordination proof networks is sure to be reached.

The results obtained with this approach suggest several lines of further research. There is the issue of introducing decay, whose impact always blurs the stable architectures[17]. It may also be interesting to further study: (i) the impact of *asymmetric*

[16] See, for instance, López et al. (2002).

[17] In a previous version of this work we considered decay and presented some preliminary results. Nevertheless, in order to have a more compact paper and to gain focus, we have chosen to leave outside this part as less conclusive.

link-constraining systems, which make sense for the one-way[18] and two-way flow models; (ii) alternative assumptions about knowledge: here we have assumed that players within each component of the link-constraining system have common knowledge of the part of the network within that component, but it may be interesting to study the effects of further restricting information, which suggests an interesting scenario for interaction between network and knowledge; (iii) the effects of heterogeneity combined with constraints. Finally, it could be interesting to see the impact of linking constraints on Jackson and Wolinsky's (1996) model and variants of it based on pairwise stability.

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